Critical Thinking in Pumping Test Interpretation

Foundations of pumping test interpretation:
1. The Theis model

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Overview

In these notes, a detailed discussion of the Theis model of an aquifer response to pumping is presented, along with the analyses that are based upon it. The Theis model is the foundation of pumping test interpretation. When applied appropriately, the Theis model yields representative estimates of the bulk-average transmissivity of a formation. The Theis model incorporates a set of relatively restrictive assumptions. Although some of these assumptions may be violated to varying degrees during actual tests, the Theis model has enduring value as a benchmark against which the observed responses to pumping can be assessed, supporting the diagnosis of site conditions.

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2. The mathematics of the Theis solution
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Additional readings

- Theis (1935)
- Cooper and Jacob (1946)

1. The Theis (1935) conceptual model

The conceptual model that underlies the Theis (1935) solution is the foundation on which all other analytical models of aquifer response to pumping are built. The Theis conceptual model is illustrated schematically in Figure 1.

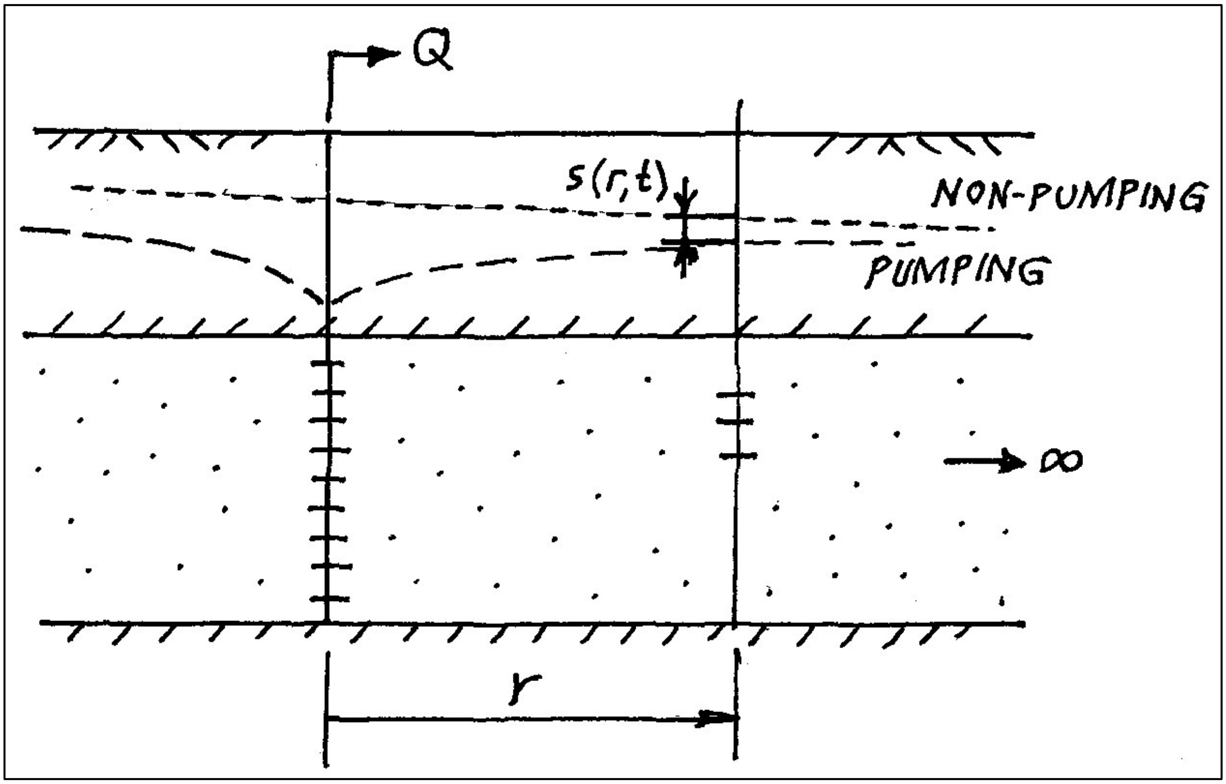


Figure 1. Conceptual model of the Theis solution

To examine the foundations of the Theis model it is essential that its underlying assumptions be established clearly. Important assumptions are made regarding the aquifer, the pumping well and observation wells and background conditions. The assumptions incorporated in the Theis conceptual model are listed on Table 1. The assumptions are assembled into four categories.

Table 1. Assumptions incorporated in the Theis conceptual model

The aquifer	
1	Darcy's Law is valid (laminar flow)
3	The transmissivity of the aquifer is uniform and isotropic
3	The aquifer is infinite in areal extent
4	The aquifer is perfectly confined by impermeable strata across its top and bottom
5	The potentiometric surface always remains above the top of the aquifer
6	The release of water from storage is instantaneous and governed by linear constitutive relations with uniform properties that remain constant through time
The pumping well	
7	There is a single pumping well
8	The pumping well penetrates the full thickness of the aquifer
9	The pumping well has an infinitesimal diameter
10	The well pumps at a constant rate
The observation wells	
11	The observation wells have infinitesimal diameters
Background conditions	
12	The changes in water levels caused by pumped have been isolated from any background temporal trends

The last assumption does not mean that the groundwater levels must be the same everywhere at the start of pumping or that they be steady prior to the test (i.e., a flat potentiometric surface). Nor does it mean that there cannot be changes in water levels during the test that are not caused by pumping from the test well. Rather, the assumption requires that the changes in groundwater levels attributable solely to pumping be established.

Motivation for considering the Theis model

The Theis model is clearly a highly idealized model of aquifer response to pumping. As practitioners, we must recognize that many of the underlying assumptions will be violated to varying degrees during actual tests. Since no actual situation will ever conform exactly to the idealized Theis model, why do we bother even invoking the model to interpret pumping test data? There are at least two good reasons:

1. The Theis solution can in fact be used to interpret at least a portion of almost all pumping test data. Although the underlying assumptions of the Theis model are quite restrictive, there is generally a portion of the test response for which the assumptions are not violated too severely; and

2. The Theis model provides us with a benchmark against which we can assess the observed responses to pumping and diagnose the actual conditions at our site. In essence, checking site conditions against a list of the ideal assumptions allows us to identify the conceptual model that best describes our own site.

Application of the Theis model provides preliminary quantitative characterization of a site, and just as importantly, provides a starting point for the diagnosis of site conditions.

2. The mathematics of the Theis (1935) solution

Starting from the assumptions listed in Section 1, the governing equation for transient radial flow to a well is written as:

$$S\frac{\partial s}{\partial t} = T\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) \qquad 0 < r < \infty$$

Here:

s: drawdown [L];
r: radial distance from the pumped well to the observation well [L];
t: elapsed time since the start of pumping [T];
T: transmissivity (= $K_H \times B$) [L^2T^{-1}];
S: storage coefficient (= $S_s \times B$) [-];
K_H: horizontal hydraulic conductivity [LT^{-1}];
S_s: specific storage [L^{-1}]; and
B: aquifer thickness [L].

The inside and outside boundary conditions are:

$$\lim_{r\to 0} 2\pi Tr\frac{\partial s}{\partial r}(r,t) = -Q$$

$$s(\infty, t) = 0$$

The initial conditions are:

$$s(r, 0) = 0$$

The parameter Q denotes the pumping rate [L^3T^{-1}]. A positive value of Q denotes a withdrawal of water from the aquifer, which gives rise to a positive drawdown, that is, a decline in water levels with respect to non-pumping conditions.

The governing equation, boundary and initial conditions comprise a classical boundary value problem. When C.V. Theis posed the problem in 1935 with respect to transient flow to a well, the solution was well known in the theory of heat conduction (for example, the solution is presented in *Introduction to the Mathematical Theory of the Conduction of Heat in Solids*, H.S. Carslaw, 1921). Theis' crucial contributions to the problem was not the solution; rather, it was his insight that an analogy could be made between the transient flow of groundwater in porous media and the transient conduction of heat in solids. Theis' analogy has formed the basis for all subsequent developments in transient groundwater hydraulics.

The solution to the boundary-value problem set up by Theis can be derived by several alternative integral transform methods or by successive integration using the Boltzmann transformation. The solution for the drawdown at any distance *r* and elapsed time *t* is written as:

$$s(r,t) = \frac{Q}{4\pi T} \int_{\frac{r^2 S}{4Tt}}^{\infty} \frac{1}{y} \, EXP\,\{-y\} \; dy$$

The integral is one version of the exponential integral, which arises in other physical applications. Abramowitz and Stegun (1972, p. 228) define the integral as:

$$\int_{x}^{\infty} \frac{1}{y} EXP\{-y\} \; dy = E_1(x)$$

The solution can be therefore be written as:

$$s(r,t) = \frac{Q}{4\pi T} E_1 \left\{ \frac{r^2 S}{4Tt} \right\}$$

The function E_1 is slightly different in form from the "classical" exponential integral defined as:

$$Ei(x) = \frac{Q}{4\pi T} \int_{-\infty}^{x} \frac{1}{y} \, EXP\,\{-y\} \; dy$$

The functions E_1 and Ei are closely related:

$$E_1(x) = -Ei(-x)$$

Using this last identity, the Theis solution can be written as:

$$s(r,t) = \frac{Q}{4\pi T} \left[-Ei \left\{ -\frac{r^2 S}{4Tt} \right\} \right]$$

The last equation is the form of the solution that usually appears in the petroleum engineering literature, where it is referred to as the "line-sink" or as the "*Ei*" solution (see for example, Matthews and Russell, 1967; p. 11).

Hydrogeologists write their version of the Theis solution as:

$$s(r,t;u) = \frac{Q}{4\pi T} W(u)$$

Here *u* is the dimensionless argument of the exponential integral:

$$u = \frac{r^2 S}{4Tt}$$

W(u) is referred to as the Theis well function.

Comparing the different forms of the Theis solution, we see that $W(u) = E_1(u) = -Ei(-u)$.

The Theis well function represents a dimensionless form, s_D, of the drawdown at any radial distance *r* and time *t*:

$$s_D = \frac{4\pi T}{Q}\, s(r,t) = W\left(u = \frac{r^2 S}{4Tt}\right)$$

The dimensionless quantity *u* is inversely related to the elapsed time, so the values of *1/u* become progressively larger as the duration of the test continues. We are accustomed to viewing time moving left to right; therefore, as shown in Figure 2 the Theis well function is generally plotted on log-log axes as *W(u)* against *1/u*.

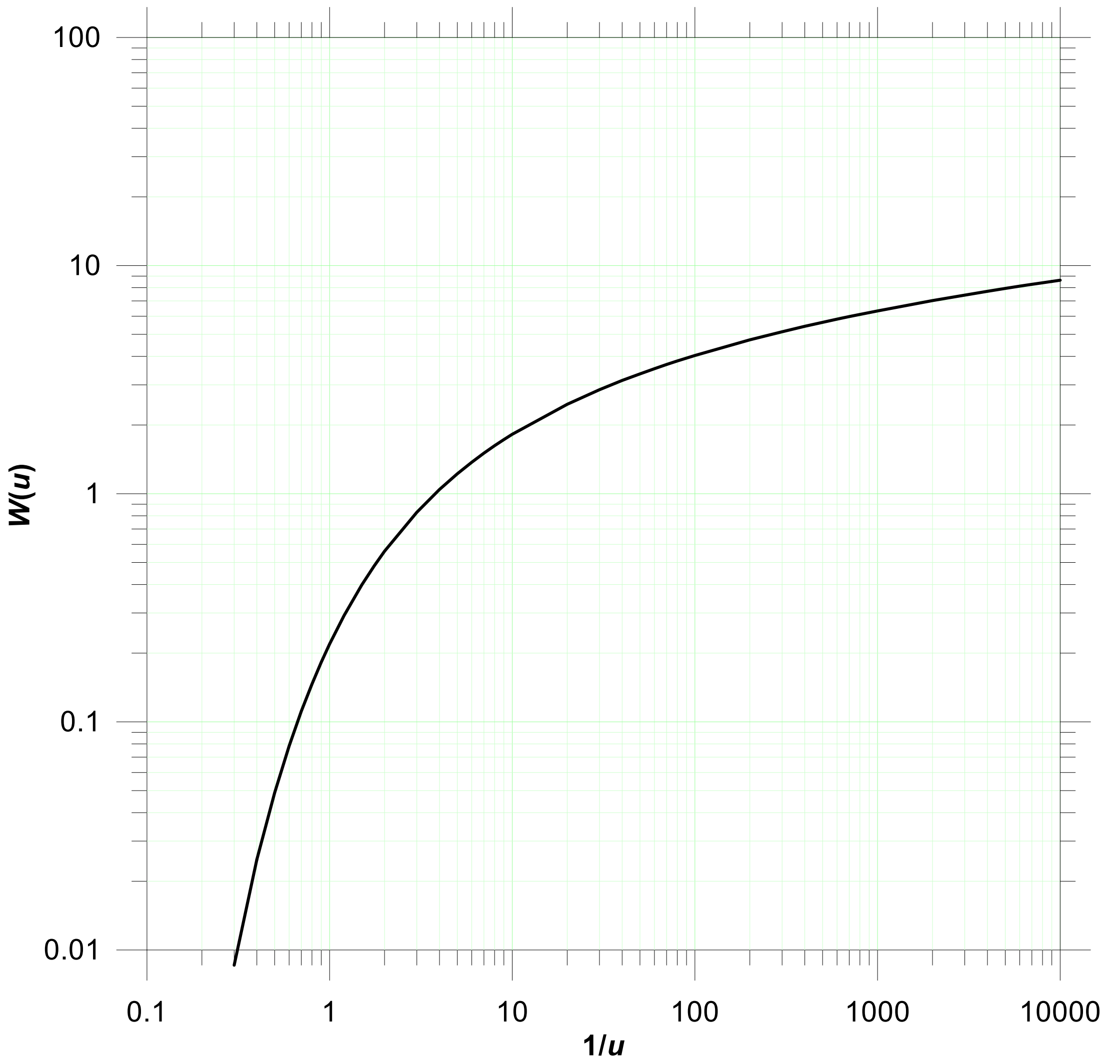


Figure 2. Theis well function

3. Analyses with the Theis (1935) solution

The transmissivity and storativity can be estimated by matching the Theis solution to the observed drawdowns with one of three ways:

- Using a type-curve matching procedure, which involves overlaying the “type curve” of Figure 2 on a log-log plot of the observations with the same scales;
- Conducting a visual match with a computer-assisted interpretation package, which involves moving the type curve with a mouse on a computer screen to overlie the data; or
- Conducting an “automatic” match with a computer-assisted interpretation package, which involves inferring the aquifer parameters from a nonlinear least squares regression.

The type-curve matching procedure is executed by superimposing the type curve on a plot of the observations and selecting a convenient “match point” from the two plots [(t,$s(t)$) and (u, $W(u)$)]. The match point does not need to lie on the type curve. There are two unknown, transmissivity and storativity, and in effect, the two “knowns” are the vertical and horizontal offsets of the two plots. The transmissivity is estimated from:

$$T = \frac{Q}{4\pi}\frac{W(u)}{s(t)}$$

The storativity is estimated from:

$$S = \frac{4T}{r^2}ut$$

Example analysis

Matching drawdowns with the Theis solution is illustrated with data from a test conducted at Gridley, Illinois in July 1953 (Walton, 1970). The layout of the wells, stratigraphy and screened intervals are reproduced in Figure 3. Well No. 3 was pumped for 8 hours at an average rate of 220 USgpm.

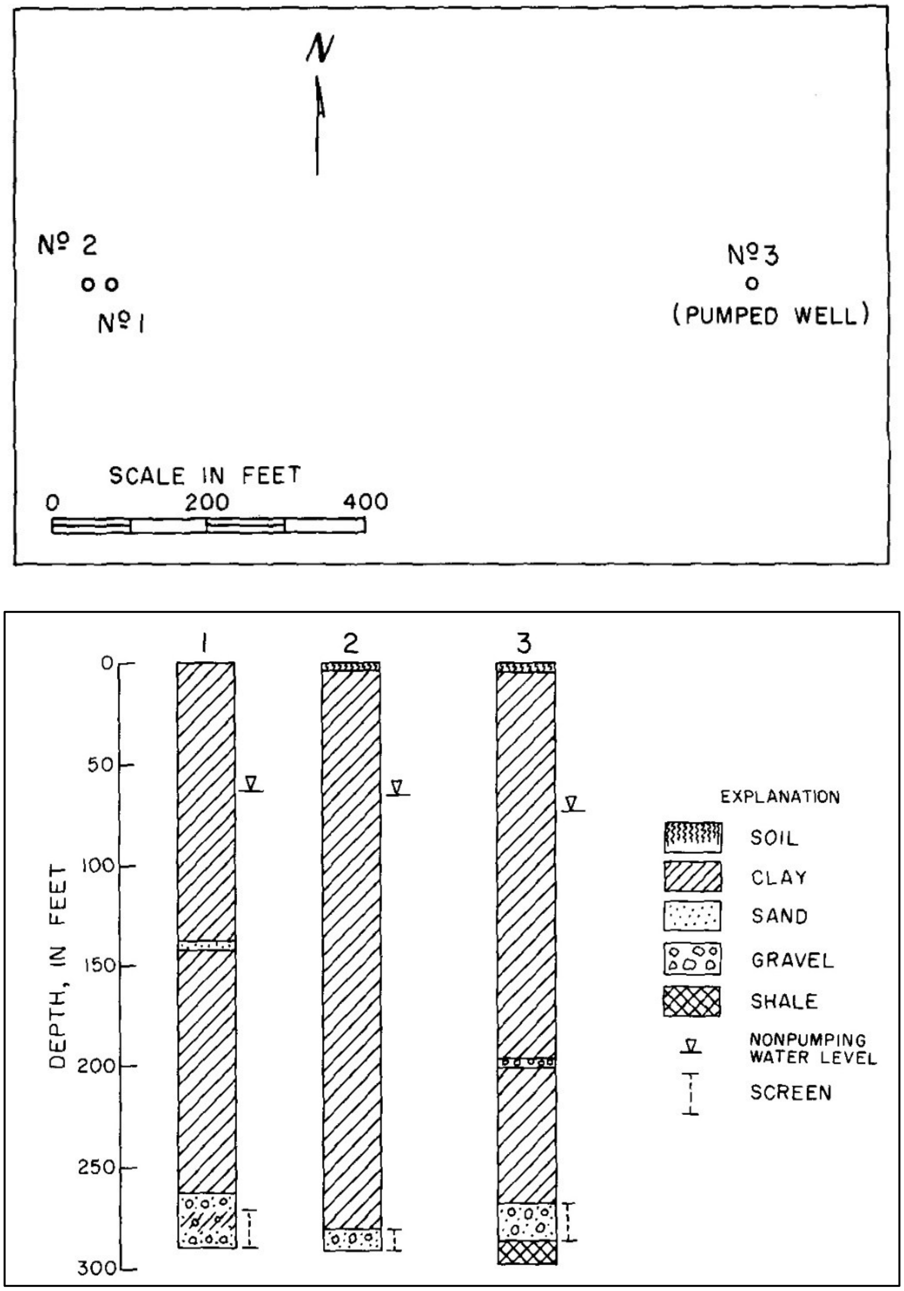


Figure 3. Gridley, Illinois pumping test layout

The drawdowns at Well No. 1, located 824 ft from the pumping well, are plotted in Figure 4.

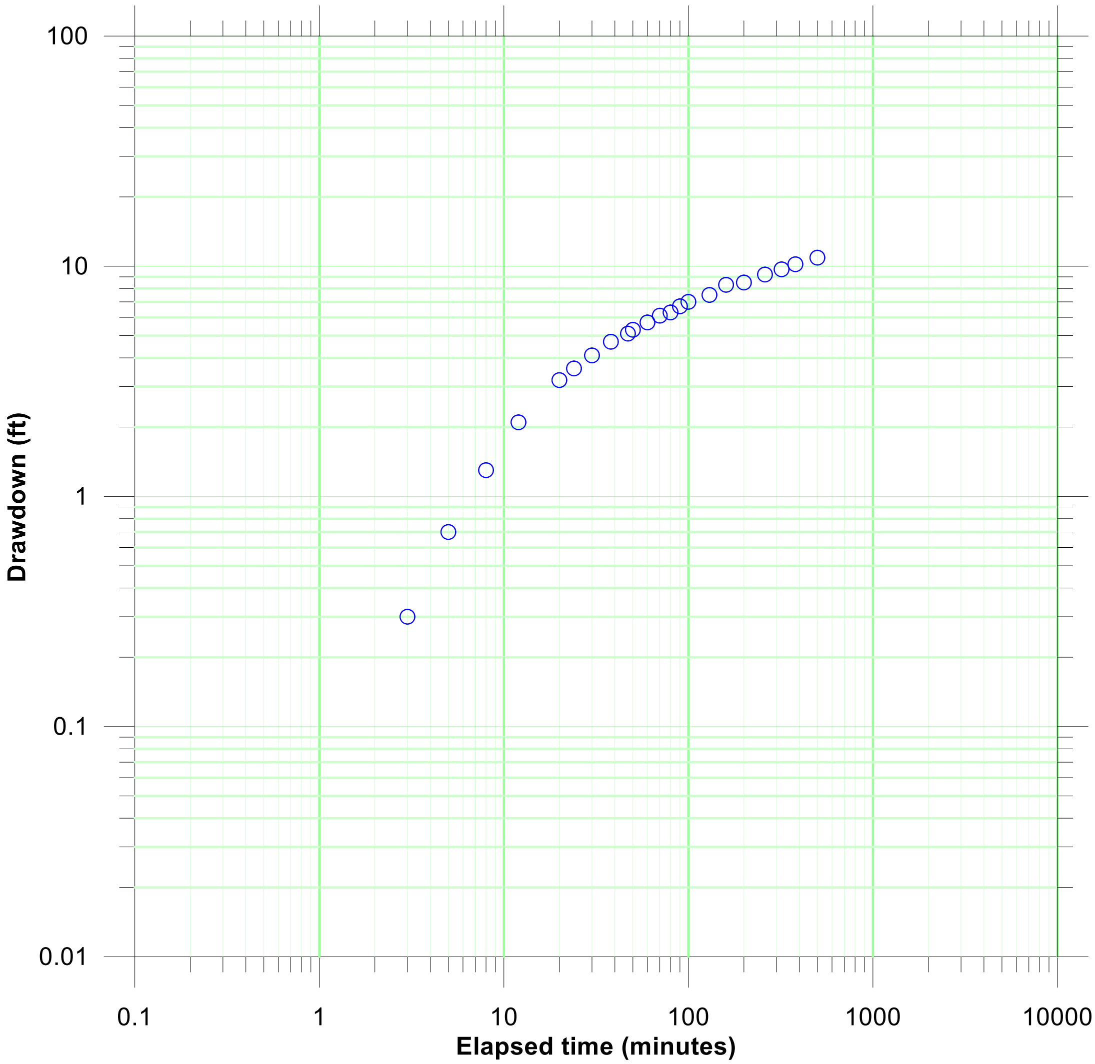


Figure 4. Drawdowns for Gridley observation Well No. 1
Data from Walton (1970)

The type curve is superimposed on the drawdown plot in Figure 5. A convenient match point on the type curve is (u = 1.0, $W(u)$ = 1.0). From the figure we estimate s = 2.5 ft at an elapsed time of 4 minutes. The transmissivity and storativity are calculated as:

$$T = \frac{\left(220 \text{ gpm} \left|\frac{192.5 \text{ ft}^3/\text{day}}{\text{gpm}}\right|\right)}{4\pi} \frac{(1.0)}{(2.55 \text{ ft})} = \mathbf{1340} \ \text{ft}^2/\text{day}$$

$$S = \frac{4(1340 \text{ ft}^2/\text{day})}{(824 \text{ ft})^2} (1.0) \left(4 \text{ min} \left|\frac{\text{day}}{1440 \text{ min}}\right|\right) = \mathbf{2.2 \times 10^{-5}}$$

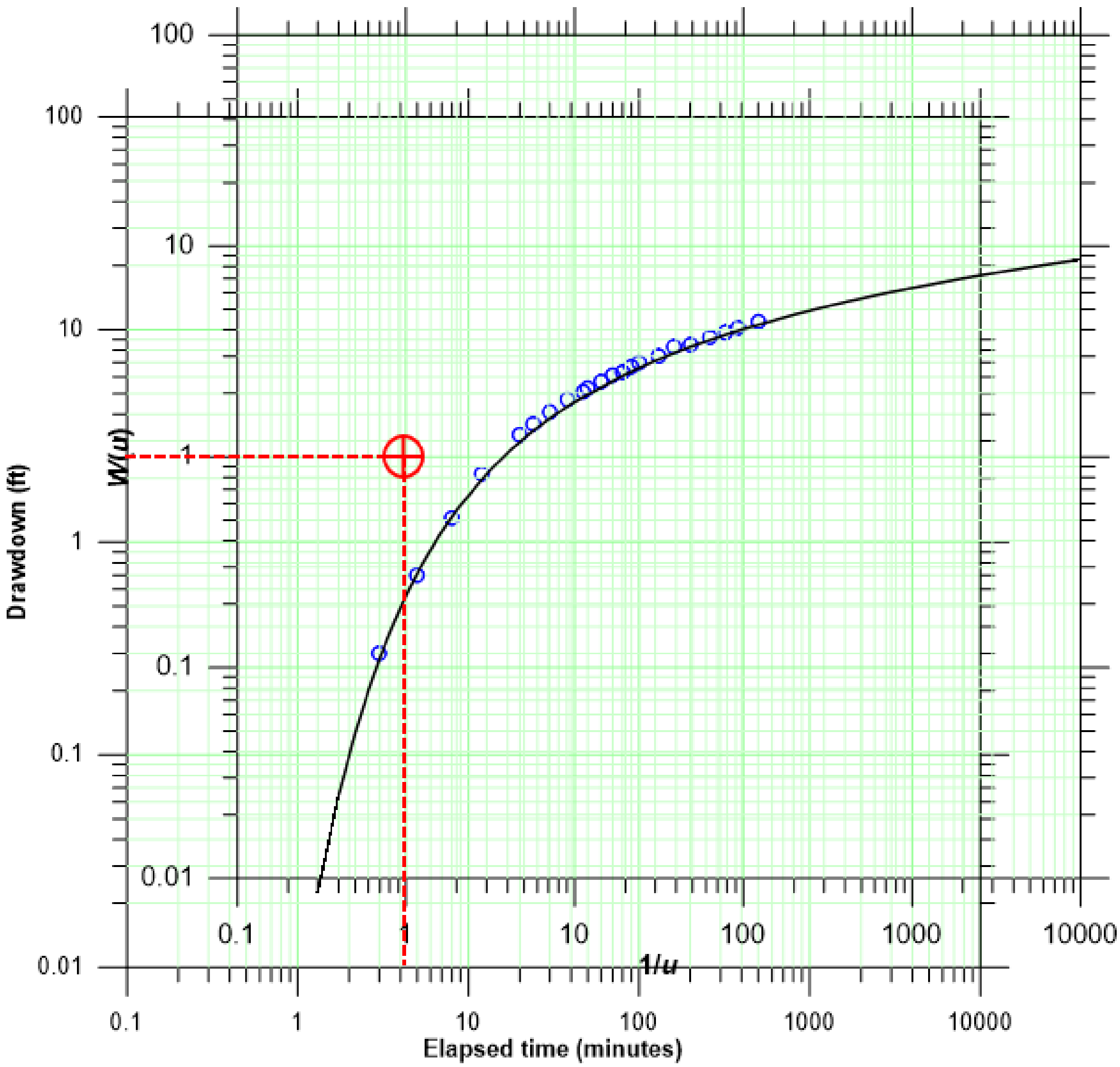


Figure 5. Gridley example, Theis type-curve match

The match to the observations obtained through a nonlinear least-squares regression fit is shown in the Figure 6. The estimated parameter values are consistent with those obtained from the type-curve match.

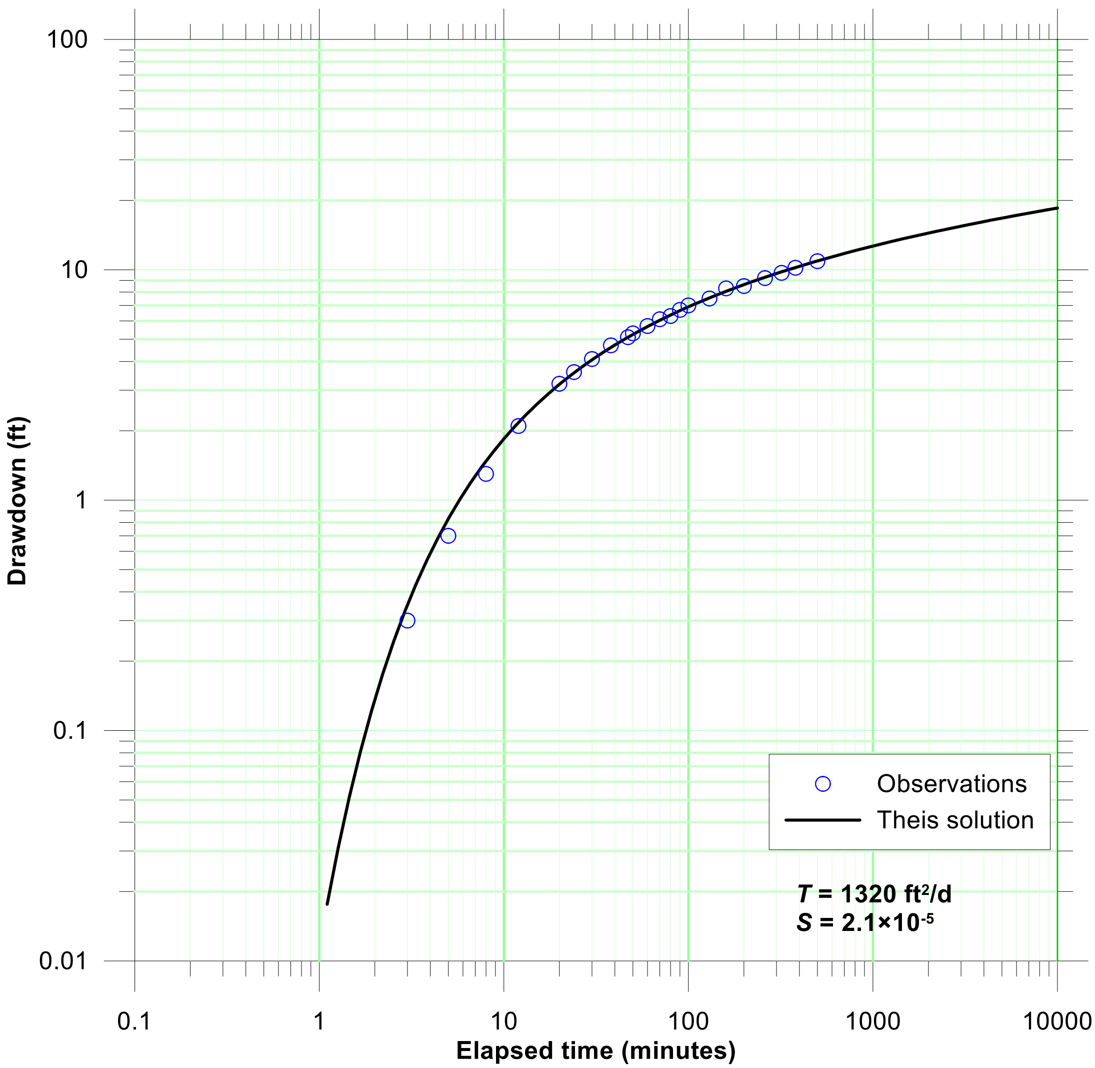


Figure 6. Match of the drawdowns with the Theis solution

4. The Cooper and Jacob (1945) approximation

The Theis well function can be expanded in the following infinite series:

$$W(u) = -0.5772 - ln\{u\} + u - \frac{u^2}{2 \times 2!} + \frac{u^3}{3 \times 3!} - \cdots$$

The leading term 0.5772 is referred to as Euler's constant.

Cooper and Jacob (1945) recognized that when *u* becomes sufficiently small, the Theis well function can be approximated closely using just the first two terms of the series:

$$W(u) \cong -0.5772 - \ln\{u\}$$

In other words, beyond some value of *u*, the arithmetic value of the dimensionless drawdown plots as a straight line against the logarithm of *u*. Reflecting this limiting behavior, the Theis well function and Cooper-Jacob approximation are shown together on a semilog plot in Figure 7. As shown in the figure, for larger values of *1/u* the Cooper-Jacob approximation matches the exponential integral closely. The limit of applicability of the Cooper-Jacob approximation is typically cited to be $u < 0.01$ ($1/u > 100$) [see for example, Todd and Mays (2005)]. However, as shown in Figure 7, the Cooper and Jacob approximation is still very close for larger values of *u*. For example, for $u = 0.1$ ($1/u = 10$), the error in the approximation is still only about 5%.

Substituting for the approximation of *W(u)* in the Theis solution yields:

$$s(r,t) = \frac{Q}{4\pi T} W(u) \cong \frac{Q}{4\pi T} [-0.5772 - ln\{u\}] = \frac{Q}{4\pi T} \left[-0.5772 - ln\left\{\frac{r^2 S}{4Tt}\right\}\right]$$

This solution can be rearranged using the properties of the log function:

$$s(r,t) \cong \frac{Q}{4\pi T} ln\left\{EXP\{-0.5772\} \times \left(\frac{4Tt}{r^2 S}\right)\right\} = \frac{Q}{4\pi T} ln\left\{2.2459 \frac{Tt}{r^2 S}\right\}$$

Finally, converting to log10 (which requires only the factor *ln*(10) = 2.303), we obtain:

$$s(r,t) \cong \frac{Q}{4\pi T} 2.303 \, log\left\{2.2459 \frac{Tt}{r^2 S}\right\}$$

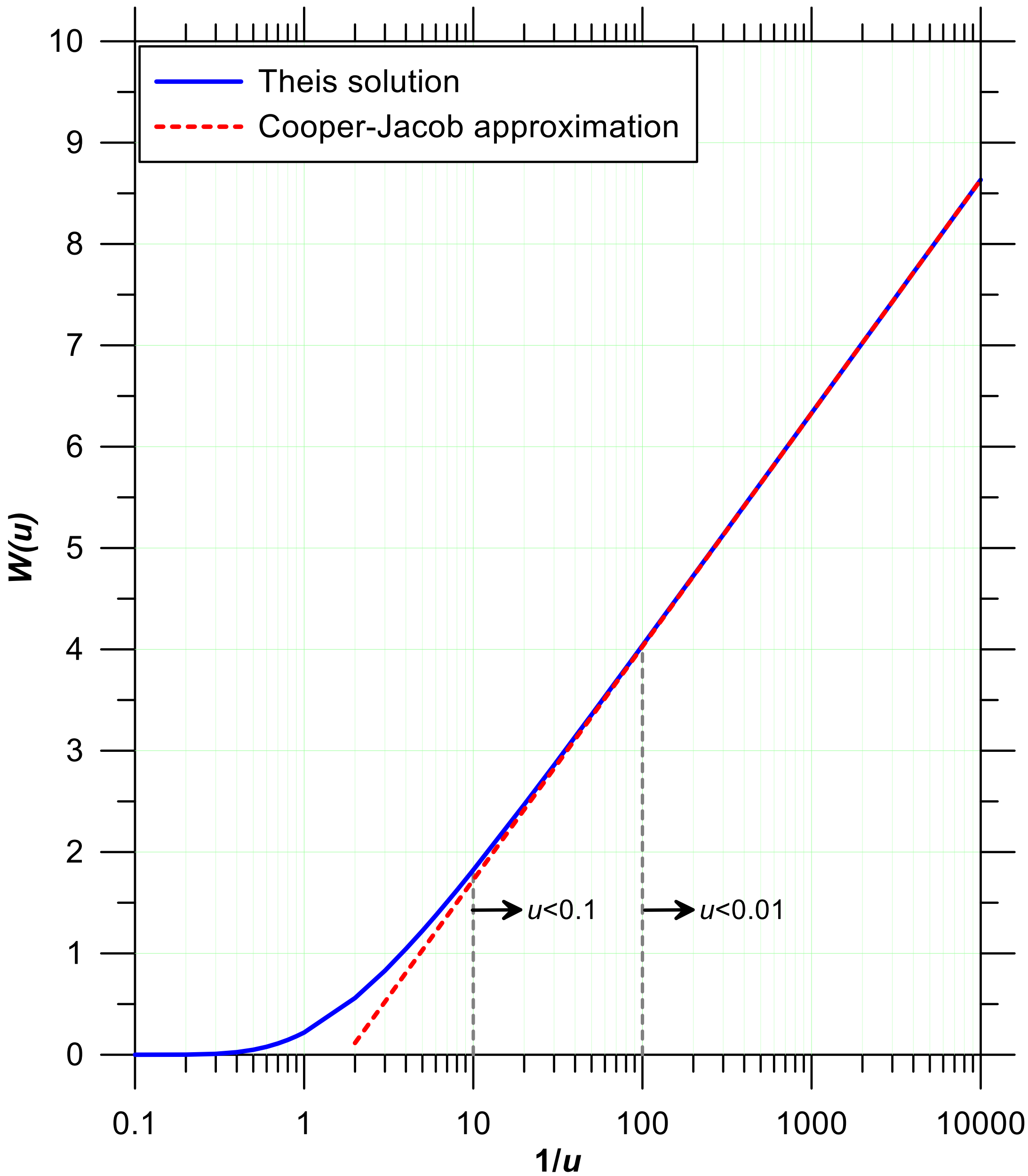


Figure 7. Cooper-Jacob approximation of the Theis well function

5. Overview of the Cooper-Jacob analyses of drawdown data

Three types of analyses are developed from the Cooper-Jacob approximation:

- Time-drawdown analysis
  The drawdown records of individual wells are analyzed, by plotting drawdown (arithmetic scale) versus elapsed time (log scale);

The interpretation of individual drawdown-versus-time records is a staple technique of hydrogeologic practice.

- Distance-drawdown analysis
  The drawdowns at multiple wells at a single time are analyzed, by plotting drawdown (arithmetic scale) versus distance from the pumping well (log scale); and

- Composite analysis
  The complete time-drawdown records of multiple wells are analyzed, with drawdown (arithmetic scale) plotted against time divided by distance-squared (t/r^2).

All three Cooper-Jacob Straight-Line (CJSL) analyses comprise three tasks:

1. Identification of that portion of the response that matches the Theis conceptual model – that is, identification of a sustained interval over which the data approximate a straight line on a semilog plot;
2. Calculation of the slope of the straight line; and
3. Estimation of the transmissivity and confined storage coefficient (storativity).

The developments of the three Cooper-Jacob analyses are presented in the following sections.

6. Cooper-Jacob time-drawdown analysis

The Cooper-Jacob time-drawdown analysis is developed by differentiating the Cooper-Jacob approximation with respect to log $\{t\}$ at a fixed radial distance *r*:

$$\left.\frac{\partial s}{\partial[\log\{t\}]}\right|_r = 2.303\ \frac{Q}{4\pi T}$$

Defining the *SLOPE*:

$$SLOPE = \frac{\partial s}{\partial(log\ t)} = \ \Delta s/\log \text{cycle}\ t$$

the transmissivity, *T*, is estimated as:

$$T = 2.303\ \frac{Q}{4\pi}\frac{1}{\Delta s}$$

The storativity is estimated by projecting the semilog straight line back to zero drawdown. The intercept along the *t* axis is denoted t_0.

$$s = 0.0 = \frac{Q}{4\pi T} 2.303\ log_{10}\left[2.2459\frac{Tt_0}{r^2 S}\right]$$

This reduces to:

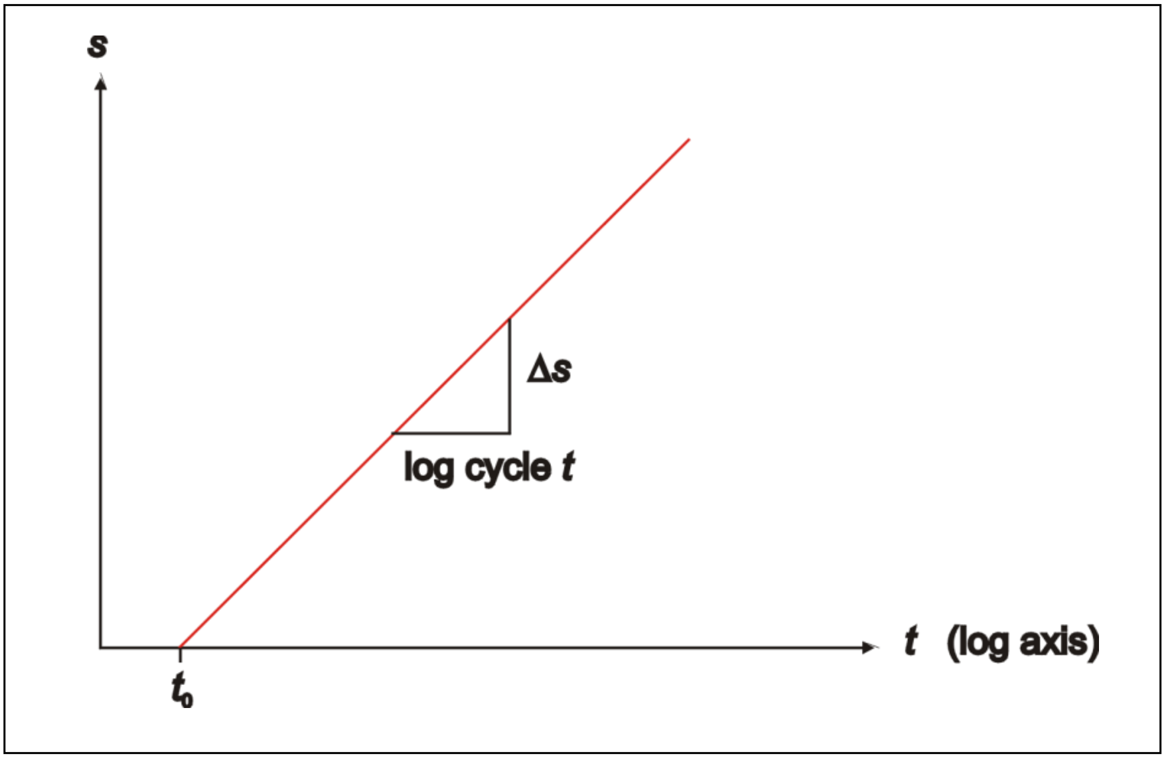
$$\left[2.2459\frac{Tt_0}{r^2 S}\right] = 1.0$$

Solving for *S*:

$$S = 2.2459\ \frac{Tt_0}{r^2}$$

Steps in the Cooper-Jacob time-drawdown analysis

1. For a well at a fixed radial distance *r*, plot *s* vs. *t* on semilog axes.



2. Estimate the slope, Δs, from the portion of the data that approximates a straight line (drawdown per log cycle *t*).

3. Estimate the intercept, t_0, by projecting the straight line back to zero drawdown.

4. Estimate the transmissivity and storativity.

$$T = 2.303 \frac{Q}{4\pi} \frac{1}{\Delta s}$$

$$S = 2.2459 \frac{T t_0}{r^2}$$

5. Assess whether the estimates of *T* and *S* and physically realistic.

Example analysis

The Cooper-Jacob time-drawdown analysis is illustrated with the data from the Gridley test analyzed previously with the Theis solution. The drawdowns at Well No. 1 are re-plotted on semilog axes in Figure 8.

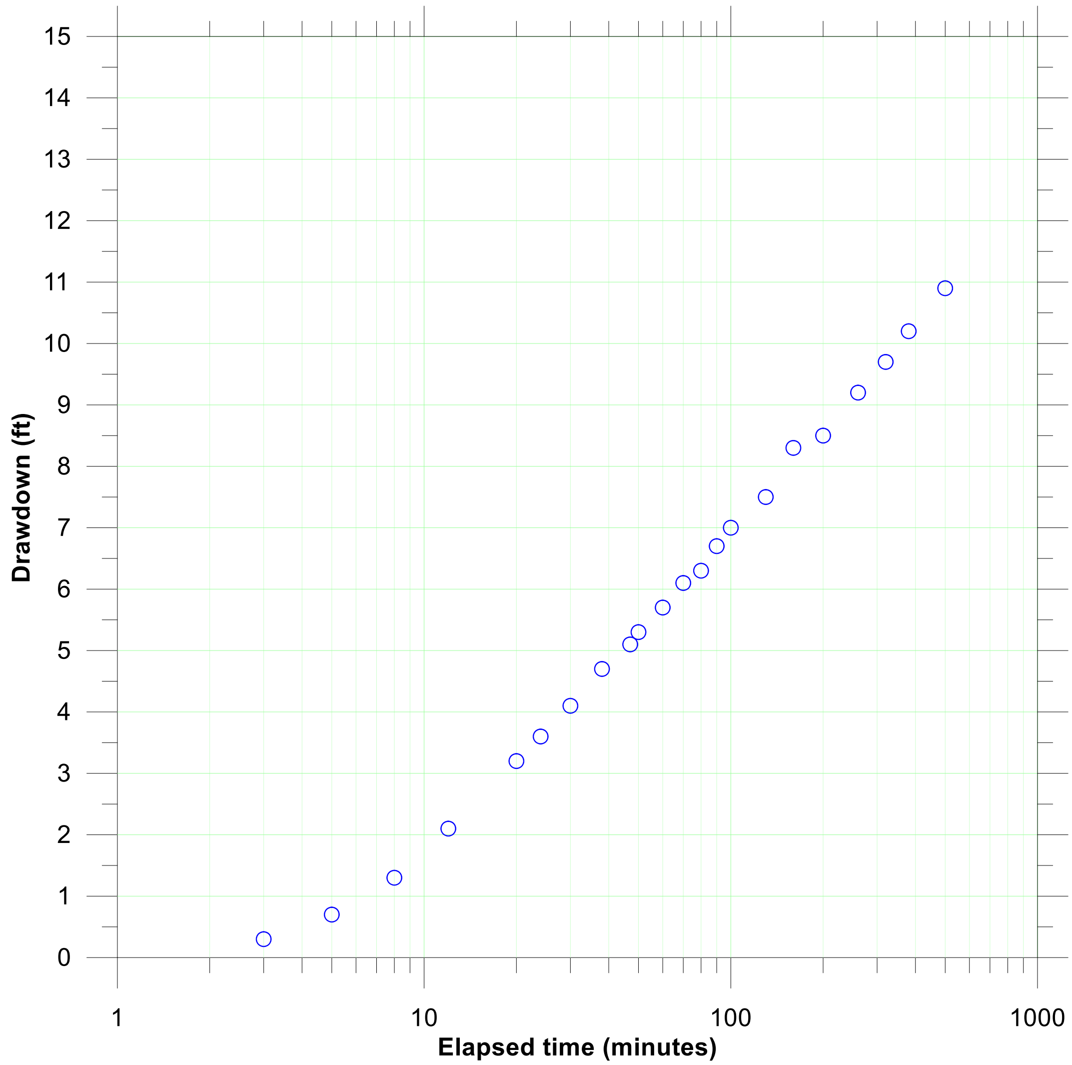


Figure 8. Drawdowns for Gridley observation Well #1
Data from Walton (1970)

Beyond about 20 minutes, the drawdowns approximate a straight line on the semilog plot. Fitting a straight line through this portion of the data yields a transmissivity of 10,500 gpd/ft:

$$T = 2.303\,\frac{(220\text{ gpm})}{4\pi}\,\frac{1}{(5.54\text{ ft})}\left|\frac{1440\text{ min}}{\text{day}}\right| = \mathbf{10,500\ \frac{gpd}{ft}}$$

Converting the transmissivity yields **1,400 ft²/d**. The estimated transmissivity is close to the value of 1,320 ft²/d estimated from the Theis analysis (Figure 6).

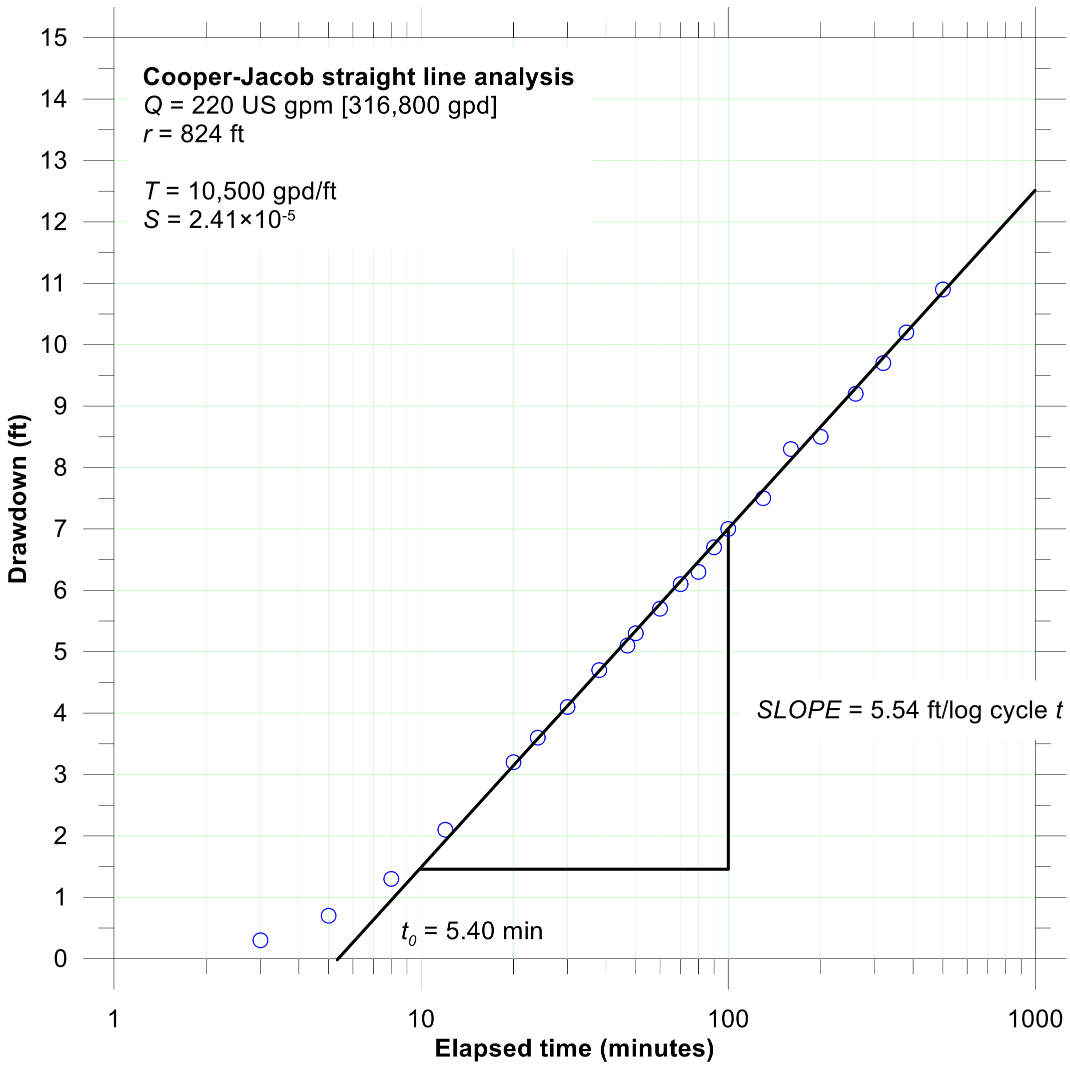


Figure 7. Cooper-Jacob time-drawdown analysis for Gridley observation Well #1

7. Motivation for using the Cooper-Jacob time-drawdown analysis

In contemporary practice, aquifer test data are generally interpreted with the aid of computer-based analysis packages. These packages support either visual type curve matching or automatic regression of the data. The obvious question arises: Why should we consider methods based on the Cooper-Jacob approximation when it is just as straightforward to use the exact Theis solution? There are at least three compelling reasons to retain and apply both analyses.

The Theis log-log analysis has a built-in threat to its reliable application. In our opinion, the log-log analysis tends to inappropriately focus the analyst's attention on the data obtained relatively soon after the start of pumping. A logarithmic axis for the drawdown visually exaggerates the magnitude of early drawdowns. Since the function *W(u)* becomes relatively flat for larger values of *1/u*, it is tempting to fit the response where the response exhibits the most distinctive curvature. As shown in Figure 8, the larger curvature occurs for the smallest values of *1/u*, that is, at relatively early times.

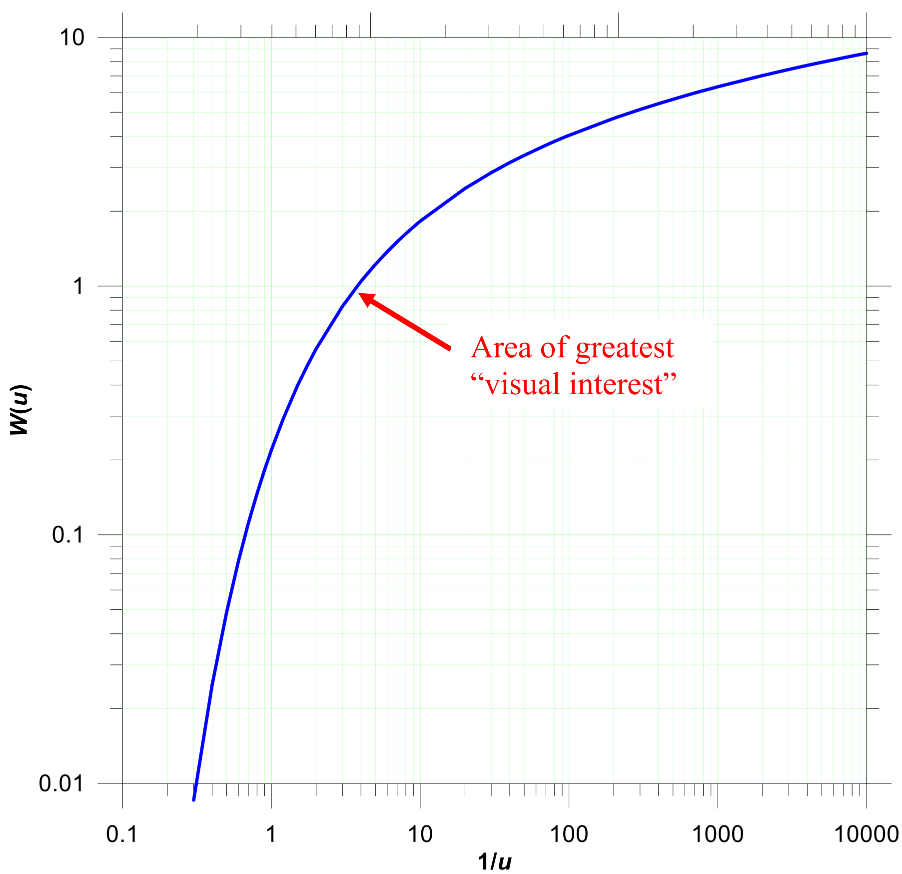


Figure 8. Area of greatest "apparent" response in the Theis solution

There is usually significant uncertainty associated with data collected soon after the start of pumping:

1. The magnitudes of the early drawdowns are relatively small, so there is bound to be some imprecision in their measurement; and

2. There is usually some noise in the data because of the adjustments in the pumping rate that are often required at the start of a test.

<u>Example 1</u>

The first source of uncertainty is illustrated with the drawdown data recorded during a pumping test conducted at Elmira, Ontario. As shown in Figure 9, this particular analysis with the Theis type-curve focused on matching the data from the first 10 minutes of pumping. These data are much less significant than the later drawdowns with respect to the diagnosis of the response of the aquifer to pumping. Furthermore, the early data are limited in their resolution. The drawdown record exhibits an interesting feature that is characteristic of data obtained with a pressure transducer: the record shows distinct jumps. The jumps are in fact directly related to the sensitivity of the transducer, rather than indicating steps in the actual response or variations in the pumping rate. The jumps make up most of the changes in water levels at the early stages of the test.

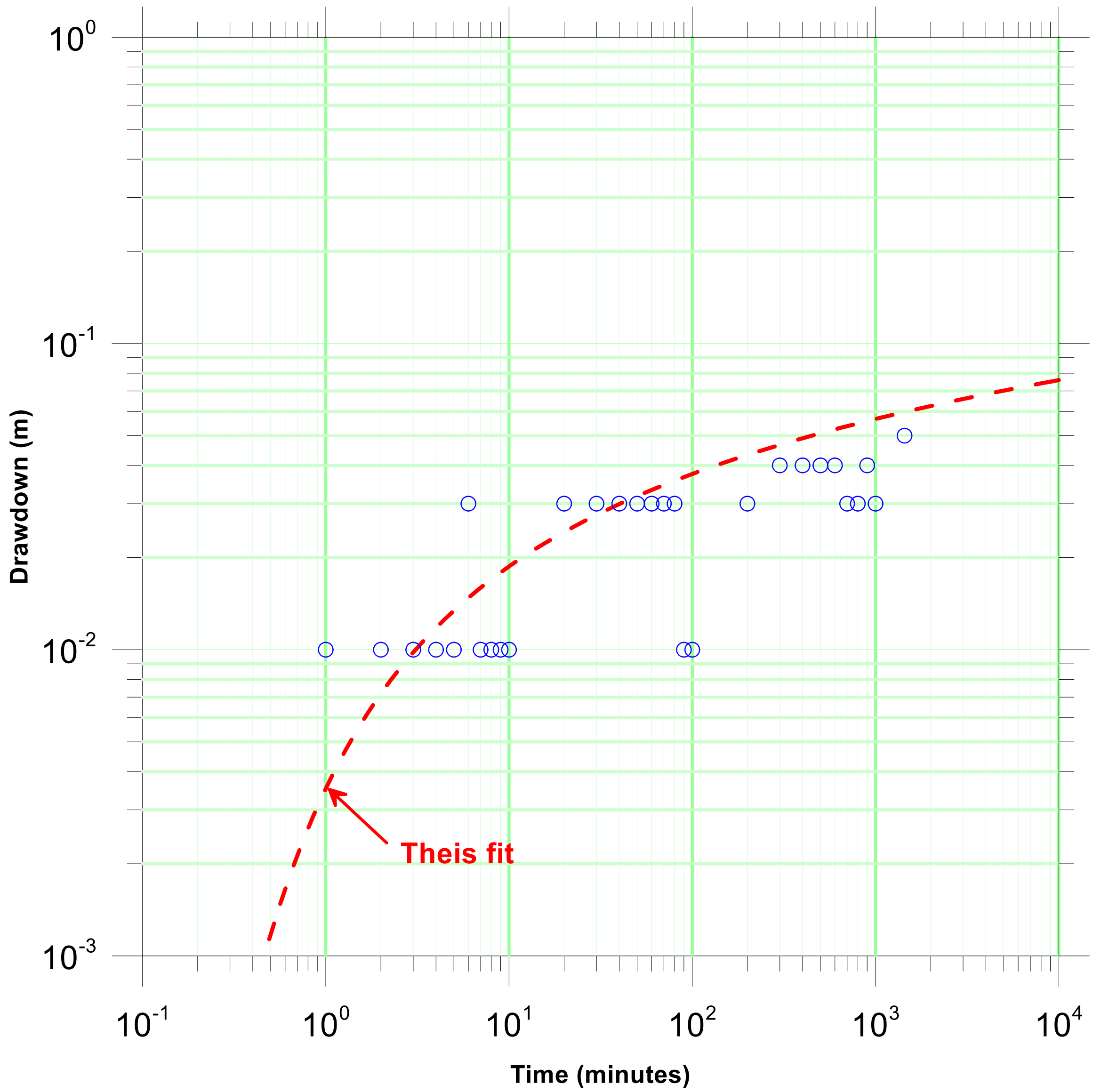


Figure 9. Drawdown response relatively distant from the pumping well

Example 2

Uncertainties introduced by brief variations in the pumping rate at the start of a test are illustrated by a hypothetical example that is intended to represent typical conditions during a pumping test. The aquifer is 5 m thick, with a horizontal hydraulic conductivity of 10^{-5} m/sec and a specific storage of 10^{-5} m^{-1}. The aquifer is pumped for 100,000 seconds (just over 1 day). It is assumed that the pumping rate is held at a constant rate of about 10 USgpm, except for a brief period of adjustment during the first 10 minutes. The pumping history is tabulated below and plotted in Figure 10.

Time (minutes)	**Pumping rate (USgpm)**	**Pumping rate (m^3/sec)**
0-5	19	1.210E-3
5-10	14	9.075E-4
10→	10	6.309E-4

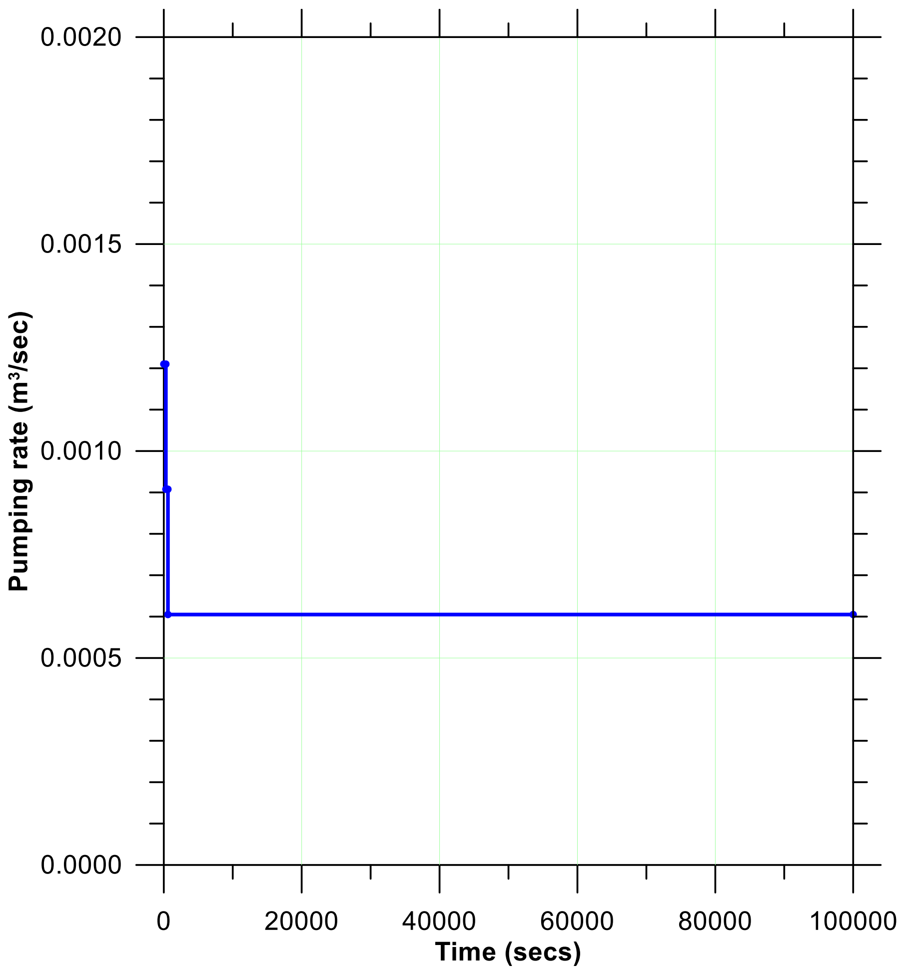


Figure 10. Pumping history for Example 2

We note two aspects of the variations in the pumping rate during the test:

1. The variations in the pumping rate considered in this example are typical for an actual test. Most tests require a brief interval during which valves are adjusted to establish a constant rate over the longer term; and
2. The duration of the period of adjustments in the pumping rate at the start of this example are brief relative to the entire duration of the test. When plotted to full scale, the two steps at the start appear as small blips. We might not even detect or record these "blips".

We will examine the drawdown at an observation well 5 m from the pumping well. The first plot of the drawdowns at the observation well, Figure 11, is made with log-log axes, in anticipation of an analysis with the Theis solution.

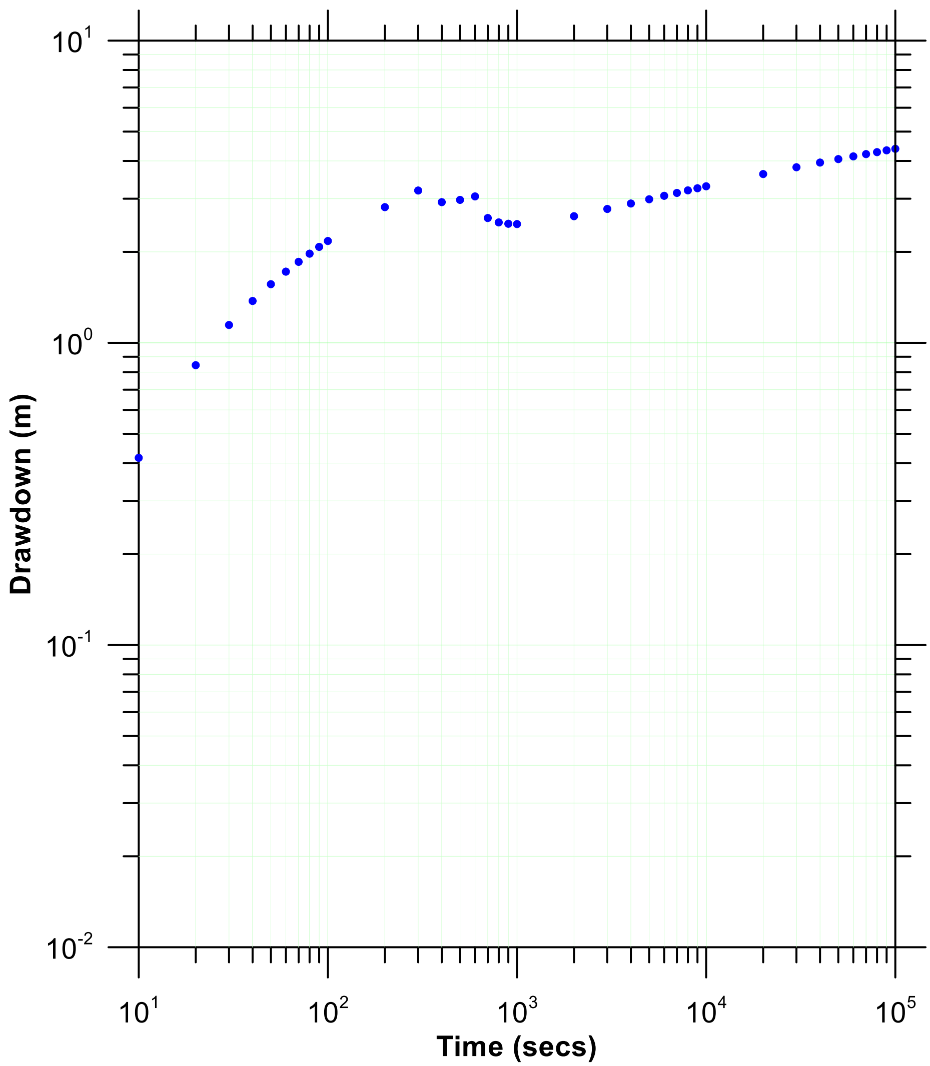


Figure 11. Drawdown calculated at *r* = 5 m

In a “real-world” situation, it is possible that the very early adjustments of the pumping rate might not even be recorded. In that case, we might be inclined to match the early portion of the data, which is indeed matched closely with the Theis type curve. The deviation between the Theis solution and the data at later times might be explained as a “boundary effect”, for example. The results of the analysis are shown in Figure 12. The estimated transmissivity is 5×10^{-5} m^2/s, half of the true transmissivity in this example. If we concentrated on the early time data but used the pumping rate that was maintained through almost the entire test, we could make very serious errors in our interpretations.

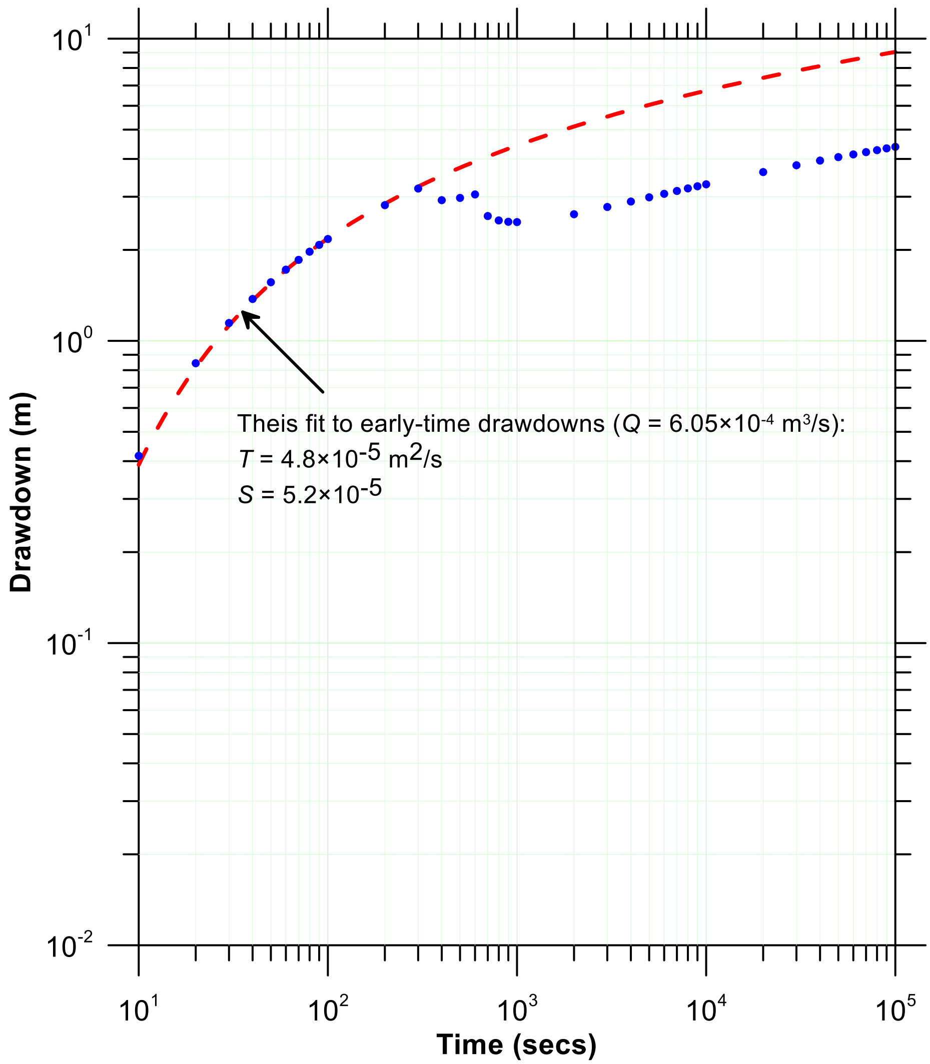


Figure 12. Match with the Theis solution: Analysis #1

The Theis solution calculated assuming a constant pumping rate of 6.309×10^{-4} m^3/s, the rate for all but the first 10 minutes of the test, is shown in Figure 13. The estimated parameters are identical to those specified for the example. If we chose to fit a Theis curve to the data beyond 1,000 seconds (which is still only 16 minutes into the day-long test), and assumed the constant rate established after 10 minutes, our interpretation is reliable; however, we open ourselves to the accusation that we have deliberately ignored much of our data.

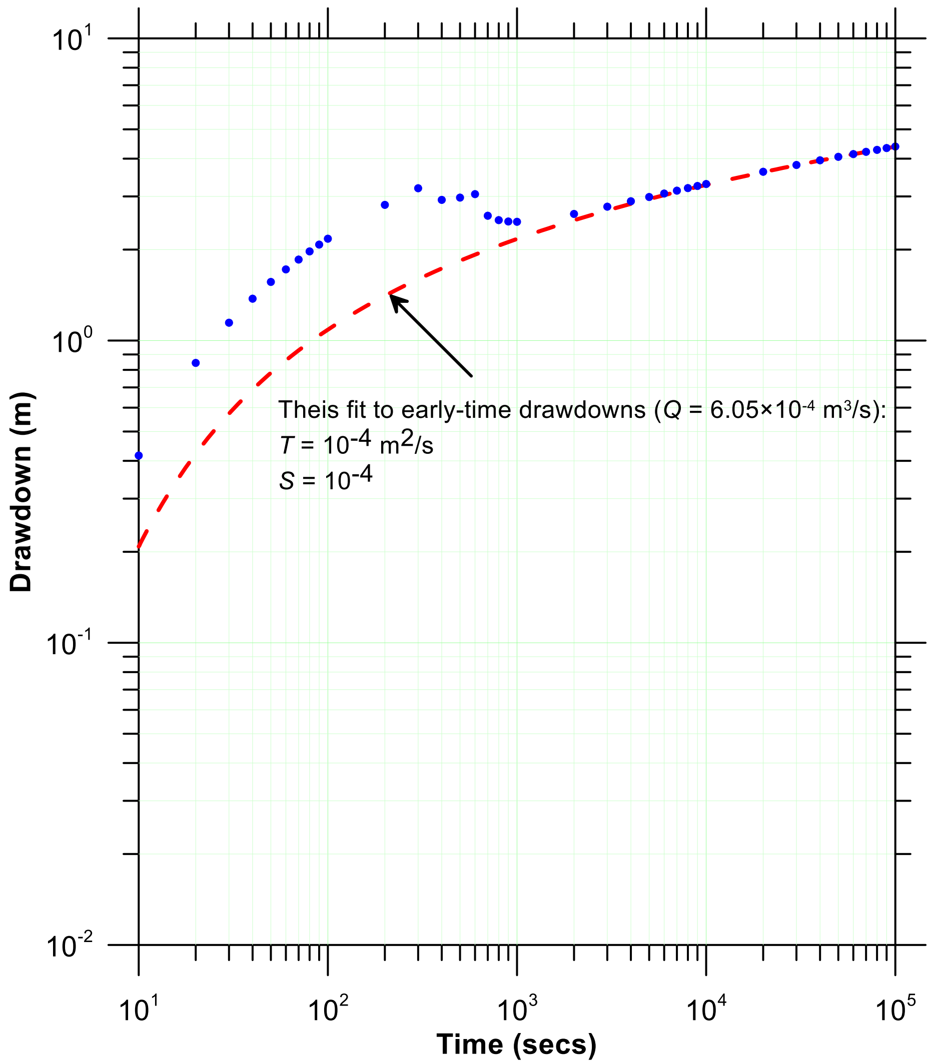


Figure 13. Match with the Theis solution: Analysis #2

The early variations in the pumping rate appear to have a dramatic effect on the response shown in Figure 13. The significance of these variations is blown out of proportion by the use of a logarithmic axis for drawdown.

The semilog plot of the drawdowns at the observation well is shown in Figure 14. This plot allows us to clearly see the evolution of the longer-term response, and thereby identify the effects of the brief early variations in the pumping rate. The dashed line shown on the plot represents the drawdown that would have been observed if the pumping rate had remained constant throughout the test. A Cooper-Jacob interpretation of the later time data is essentially insensitive to the early variations in the pumping rate.

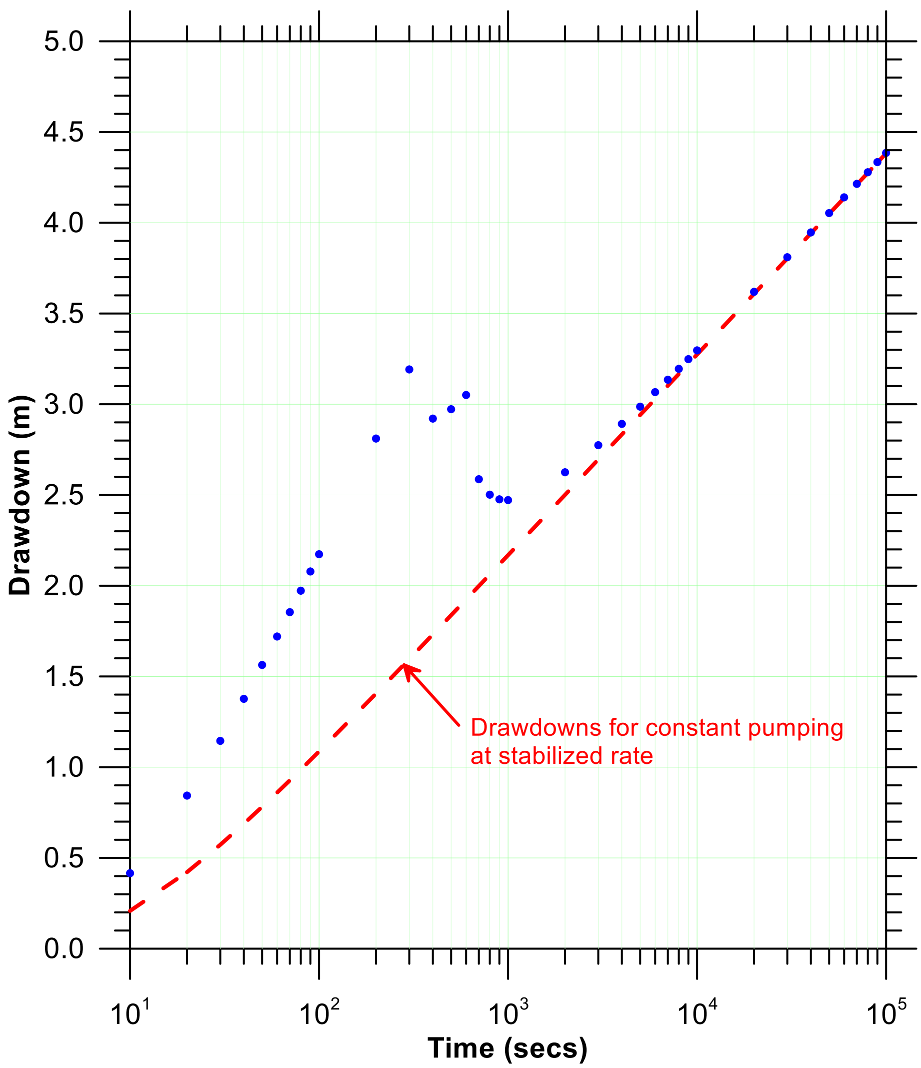


Figure 14. Drawdown calculated at *r* = 5 m, semilog plot

8. Choosing between the Theis and Cooper-Jacob analyses

The preceding examples suggest that analyses that focus on the earliest portion of the drawdown response may yield unreliable estimates of aquifer properties. The examples may have suggested that the Theis log-log analysis is prone to this problem. It is important to note that this is not a fundamental defect in the Theis analysis. Rather, these examples should be taken as warnings that like all analysis techniques, the Theis log-log analysis must be applied critically. Cooper-Jacob analyses are only superior if they lead the analyst to focus on the portion of the aquifer response that yields the most representative estimate of transmissivity.

The key point to bear in mind is that the conceptual models that underlie the Theis and Cooper-Jacob analyses are identical. The two analyses do not provide independent transmissivity estimates. Rather, the two analyses are complementary. Every pumping test analyst should employ both methods for the same set of data. The results from the two methods should be similar, demonstrating that the interpretation is at least internally consistent. This approach is demonstrated with a case study.

Case study

A pumping test to support a development application was conducted at a site in Kinloss Township, southern Ontario. The test was conducted in a well open across the upper portion of the bedrock. Key aspects of the test are indicated below.

- The depth to the water table on-site is between 2 and 4 m.
- The well is open in the upper portion of the bedrock, in the Dundee Formation, a dolomitic limestone. The primary water-bearing zone in the bedrock is between 2 and 4 m below the overburden-bedrock contact.
- On site, the overburden is approximately 33 m thick. The overburden consists of sand and bouldery gravel with thickness up to 27 m, underlain by a stiff, dense stony till that is referred to locally by drillers as hardpan. The presence of the till unit results in a confined bedrock aquifer.

The well was pumped for 1 day at an average rate of 25 Igpm (1.89×10^{-3} m^3/sec). Drawdowns were measured only in the 4-inch diameter pumping well (r_w = 0.0508 m). The drawdowns were analyzed using the Theis and Cooper-Jacob methods.

i. Theis analysis

The Theis analysis is reproduced in Figure 15. The application of the Theis analysis for this example appears to be a reasonable. The analyst restricted the type-curve match to between about 20 seconds and 400 seconds. The earliest drawdown may incorporate some wellbore losses, and the drawdowns appear to stabilize beyond 1000 seconds of pumping.

The transmissivity estimated with the analysis is $\underline{3.2\times10^{-5}\ m^2/sec}$.

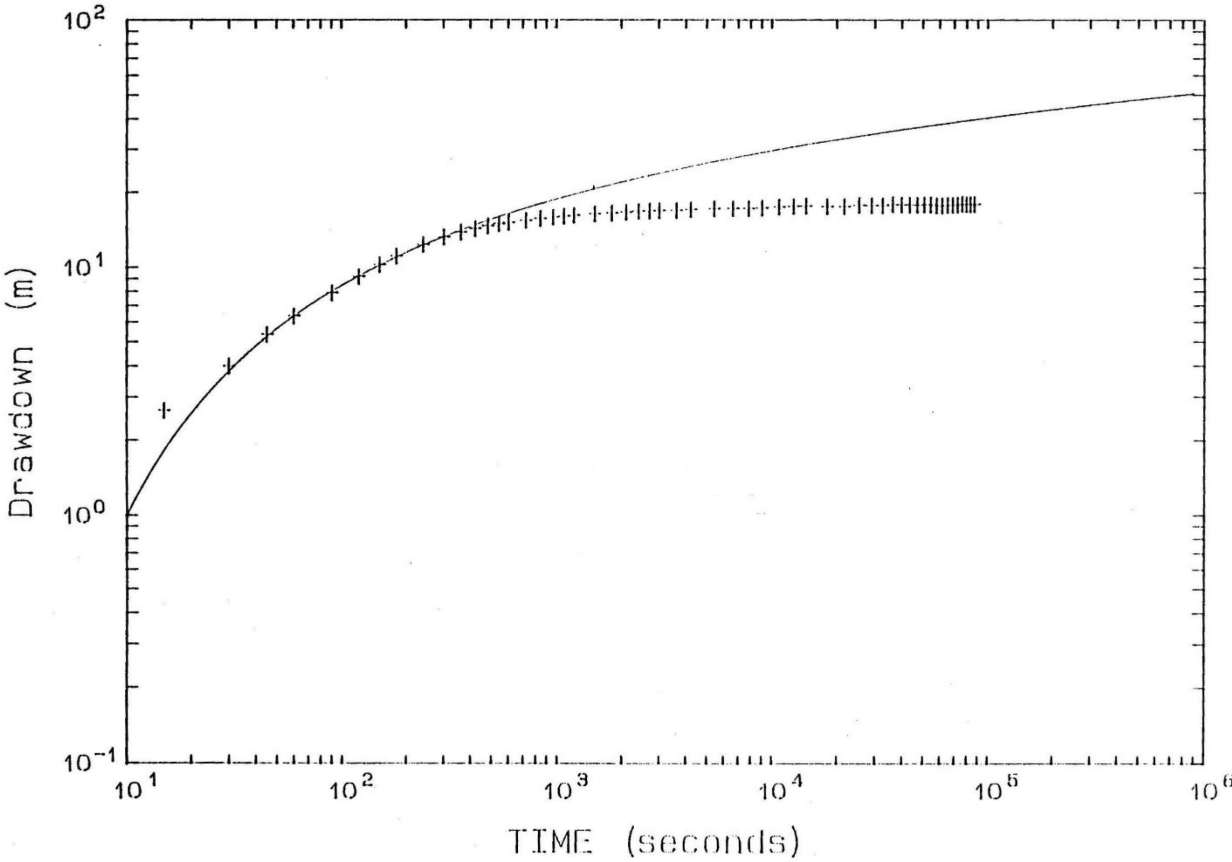


Figure 15. Pumping test example, Theis analysis

ii. Cooper-Jacob straight-line analysis

The reported Cooper-Jacob analysis is reproduced in Figure 16. The analyst identified what appeared to be a straight line and fit that portion of the response.

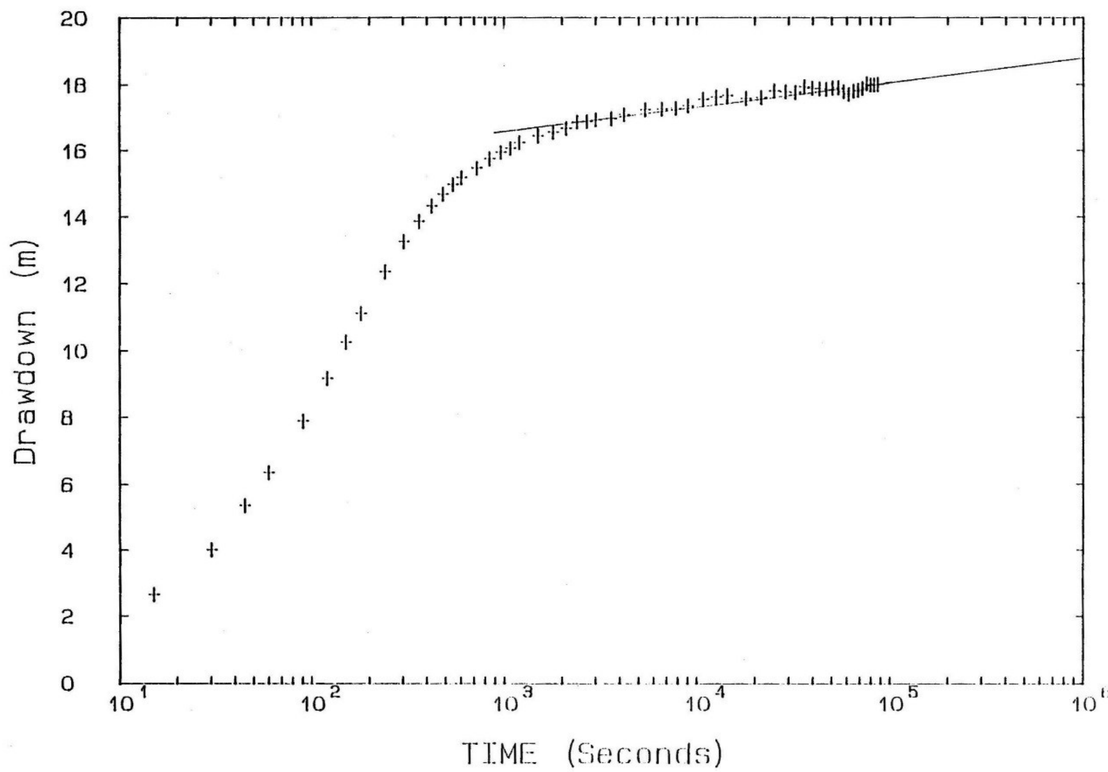


Figure 16. Pumping test example, Reported Cooper-Jacob analysis

The slope of the line drawn by the analyst is about 0.72 m per log cycle of time. If we substitute this into the formula for the transmissivity, we obtain:

$$T = 2.303\frac{(1.89 \times 10^{-3}\ \mathrm{m^3/sec})}{4\pi}\frac{1}{(0.72\ \mathrm{m})} = \mathbf{4.8 \times 10^{-4}\ m^2/sec}$$

The transmissivity estimated with the Cooper-Jacob analysis is about 15 times higher than the value estimated with the Theis analysis. A qualified hydrogeologist would not simply report both transmissivity estimates and invite the reader to choose the more reliable value. Rather, a qualified hydrogeologist would note that the Theis and Cooper-Jacob analyses share the same underlying conceptual model and would further note that internally consistent analyses should yield similar transmissivity estimates.

iii. Cooper-Jacob straight-line re-analysis

In reviewing the preceding analyses and results, we should immediately notice that the Theis and Cooper-Jacob analyses were fit to completely different portions of the drawdown data. The Theis analysis was applied to the data between 20 and 400 seconds. In contrast, the semilog straight line was fit to the data that begin at about 3,000 seconds. The data seem to fall closely on two straight lines. The question we should ask ourselves is: Was the Cooper-Jacob analysis applied to the appropriate portion of the data?

In Figure 17, the Cooper-Jacob analysis is applied to the same portion of the data as the Theis analysis. As shown in the figure, these data also approximate a straight line. However, the slope of the first line is about 10.3 m per log cycle of time. If we substitute the slope of 10.3 m per log cycle *t* into the formula for the transmissivity, we obtain:

$$T = 2.3026 \frac{(1.89 \times 10^{-3}\ \mathrm{m^3/sec})}{4\pi} \frac{1}{(10.3\ \mathrm{m})} = \mathbf{3.4 \times 10^{-5}\ m^2/sec}$$

The revised transmissivity estimate is essentially the same as the estimate obtained with the Theis analysis (3.2×10^{-5} $\mathrm{m^2/sec}$).

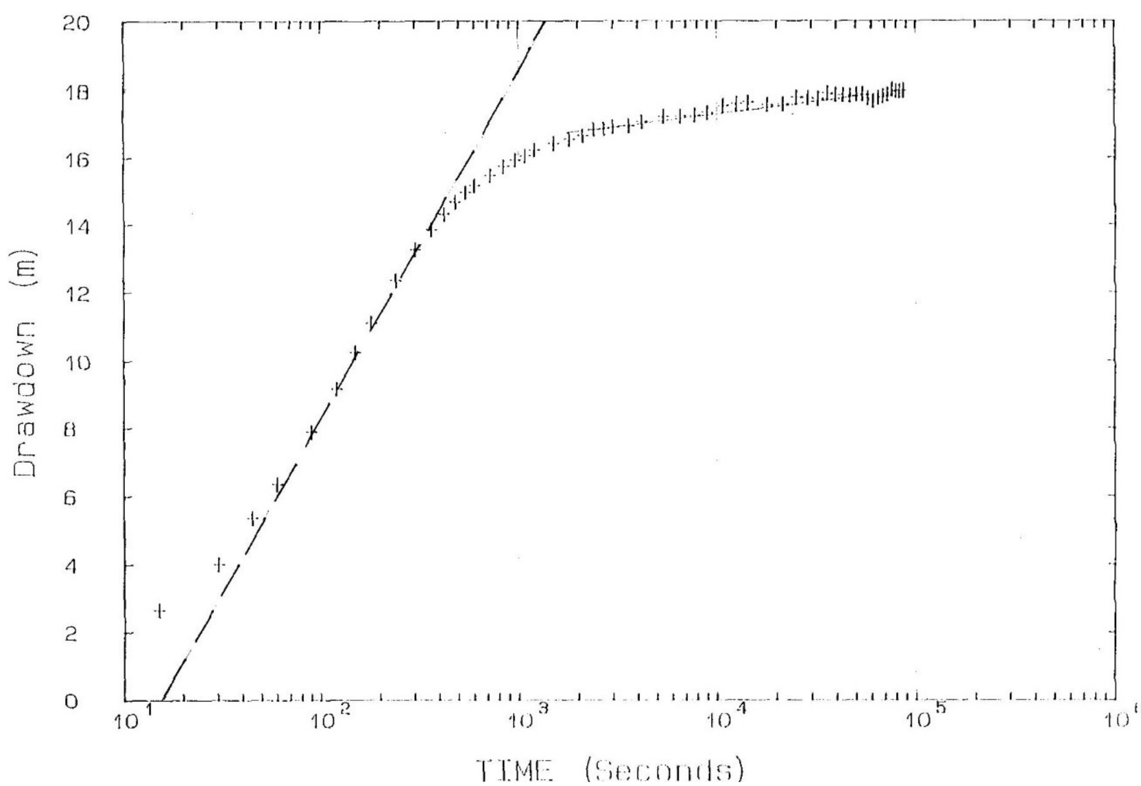


Figure 17. Pumping test example, Revised Cooper-Jacob analysis

9. Introduction to Derivative Analysis

A Cooper-Jacob straight-line analysis consists of two tasks:

1. Identification of that portion of the response that matches the Theis conceptual model – that is, identification of a sustained interval over which the data fall on a straight line in semilog space; and

2. Calculation of the slope for application in the straight-line formula.

The identification of the portion of the response that matches the Theis conceptual model is simplified by plotting the *drawdown derivative*. French petroleum engineers led by D. Bourdet pioneered this form of data treatment (Bourdet and others, 1983; Bourdet and others, 1989).

Bourdet and his co-workers defined the derivative as:

$$D_t\ (s) = \frac{\partial[s(r,t)]}{\partial[\ln\{t\}]}$$

The derivative is defined with respect to the logarithm of time, rather than time itself. This definition follows directly from the Cooper-Jacob analysis. Recalling the Theis solution:

$$s(r,t) = \frac{Q}{4\pi T} W(u) = \frac{Q}{4\pi T} \int_u^\infty \frac{e^{-x}}{x} dx$$

Substituting for the Theis solution in the definition of the derivative yields:

$$D_t\ (s) = \frac{\partial}{\partial[\ln\{t\}]} \left[\frac{Q}{4\pi T} \int_u^\infty \frac{EXP\{-x\}}{x}\ dx \right]$$

$$= \frac{Q}{4\pi T} \frac{\partial}{\partial(u)} \left[\int_u^\infty \frac{EXP\{-x\}}{x}\ dx \right] \frac{\partial u}{\partial[\ln\{t\}]}$$

$$= \frac{Q}{4\pi T}\ EXP\{-u\}$$

$$= \frac{Q}{4\pi T}\ EXP\left\{-\frac{r^2 S}{4Tt}\right\}$$

If the Theis well function is replaced with the Cooper-Jacob approximation, the derivative of the drawdown with respect to the natural log of time is:

$$D_t\ (s) = \frac{\partial}{\partial[\ln\{t\}]}\left[\frac{Q}{4\pi T}\ln\left\{2.2459\ \frac{Tt}{r^2 S}\right\}\right]$$

This is simply:

$$D_t\ (s) = \frac{Q}{4\pi T}$$

The fact that the derivative of the Cooper-Jacob approximation is a constant should not come as a surprise. The Cooper-Jacob approximation is valid when drawdown plots as a straight line against the log of time.

We recall that the slope of the straight line on the Cooper-Jacob semilog plot is:

$$\Delta s\ = 2.303\frac{Q}{4\pi T}$$

The ratio of the value of the slope and the value of the derivative when it approximates a plateau is simply $ln\{10\}$, the conversion factor between the natural logarithm and the base-10 logarithm.

Defining the dimensionless drawdown as:

$$s_D = \frac{4\pi T}{Q}s(r,t)$$

The dimensionless forms for the derivative become:

- Theis solution: $D_t(s_D) = EXP\{-u\}$; and
- Cooper-Jacob approximation: $D_t(s_D) = 1.0$

The dimensionless form of the derivative of the Theis solution is plotted in Figure 18. As expected, the derivative converges on the constant value given by the Cooper-Jacob approximation. The approach to a constant value of derivative is referred to as a "plateau" in the derivative plot.

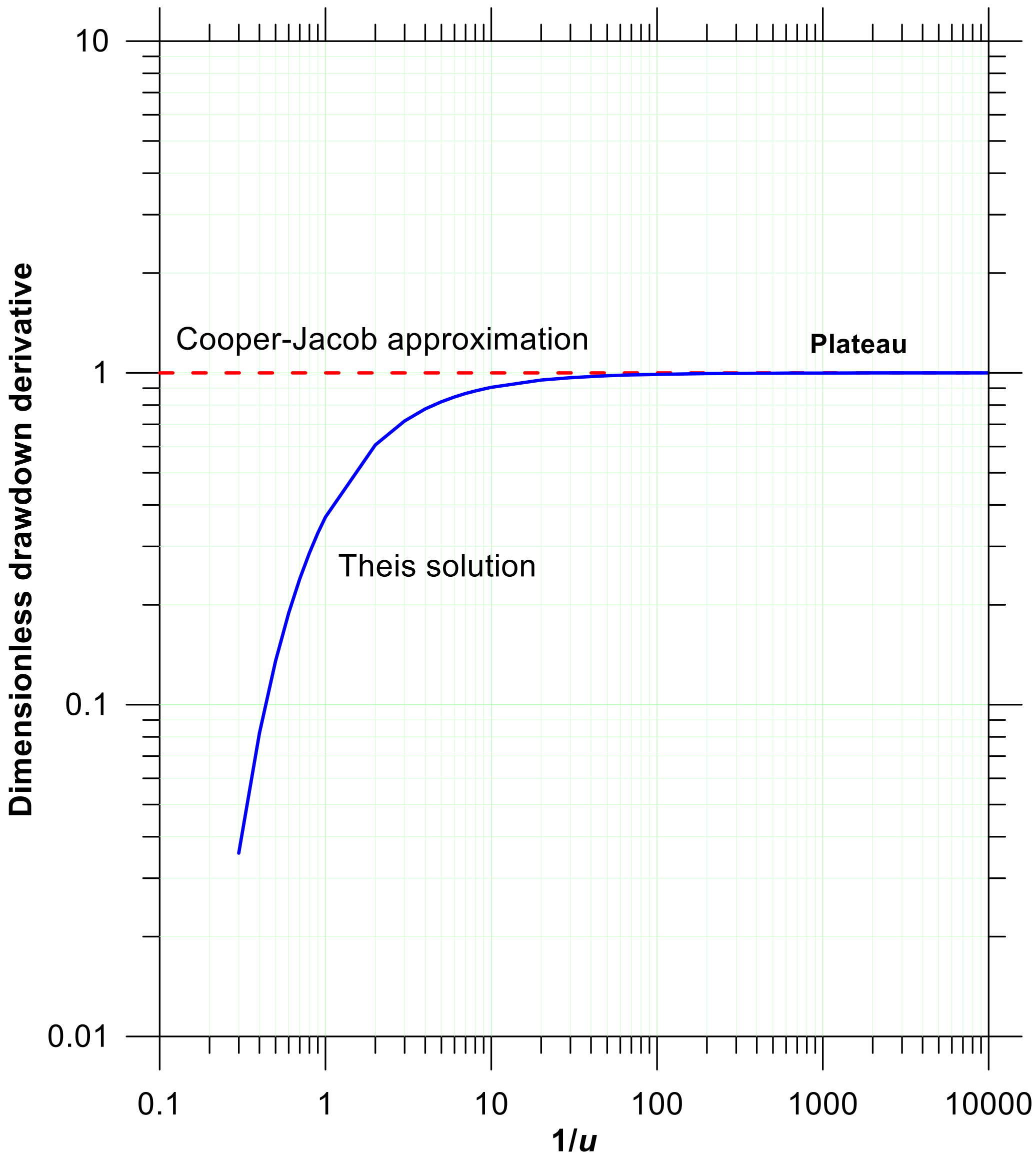


Figure 18. Dimensionless derivative of the Theis solution

Since the Cooper-Jacob approximation holds at all but the earliest times, an appropriate criterion for the interval of the response that corresponds to the Theis conceptual model is the onset of the plateau of the derivative plot (Figure 19).

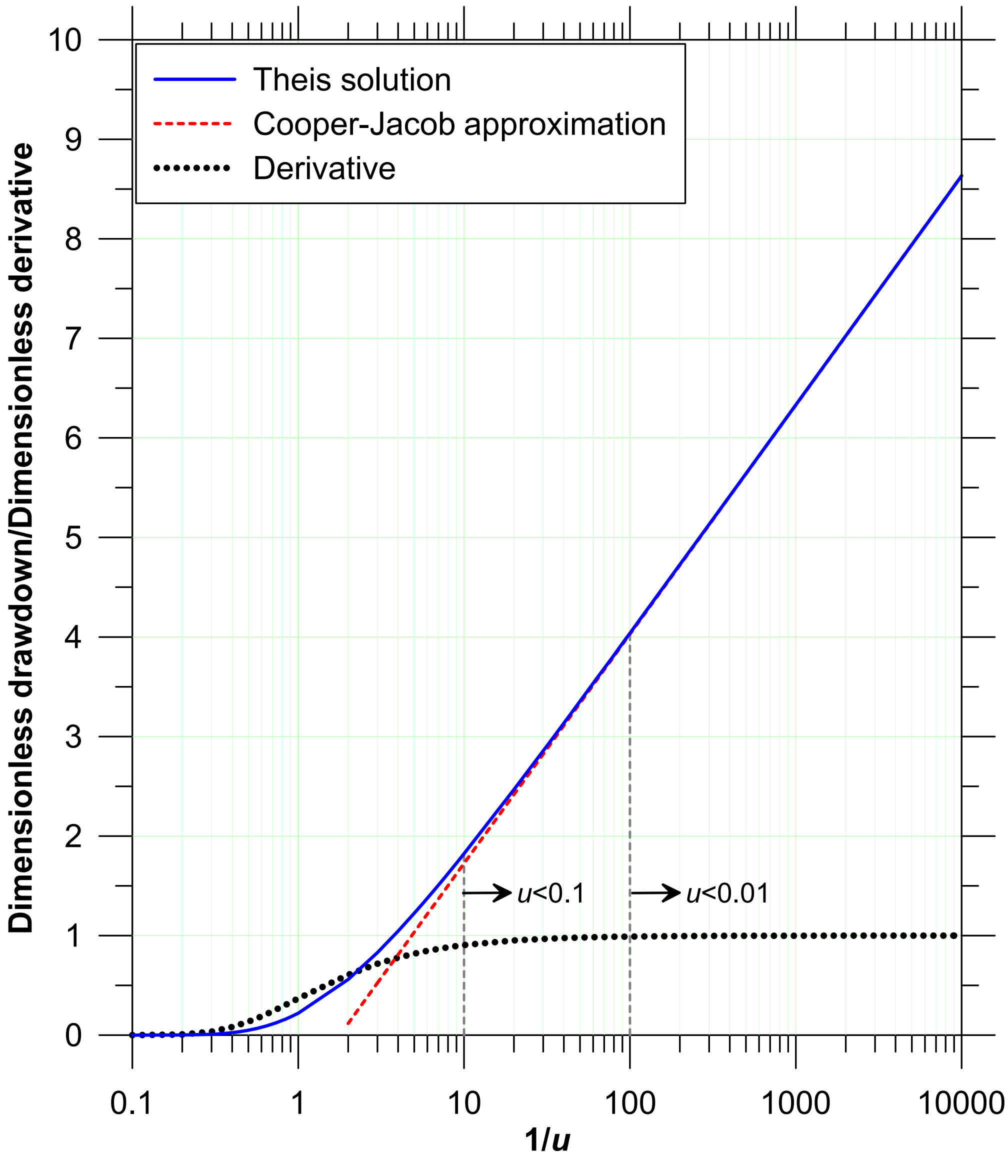


Figure 19. Cooper-Jacob analysis with the drawdown derivative

Example analysis with the derivative

The addition of the derivative to the semilog plot increases the defensibility of a Cooper-Jacob straight-line (CJSL) analysis. The application of the derivative is illustrated with data from the Gridley pumping test (Walton, 1970). As shown in Figure 20, the simultaneous plotting of the drawdown and the derivative confirms that the derivative reaches a plateau. The addition of the derivative helps identify in a direct visual manner the appropriate portion of the response for the analysis.

The transmissivity is calculated directly from the value of the plateau of the derivative:

$$T = \frac{Q}{4\pi}\frac{1}{D_t} = \frac{(220\text{ gpm})}{4\pi}\frac{1}{(2.4\text{ ft})}\left|\frac{192.5\text{ ft}^3/\text{d}}{\text{gpm}}\right| = \mathbf{1400\ ft^2/d}$$

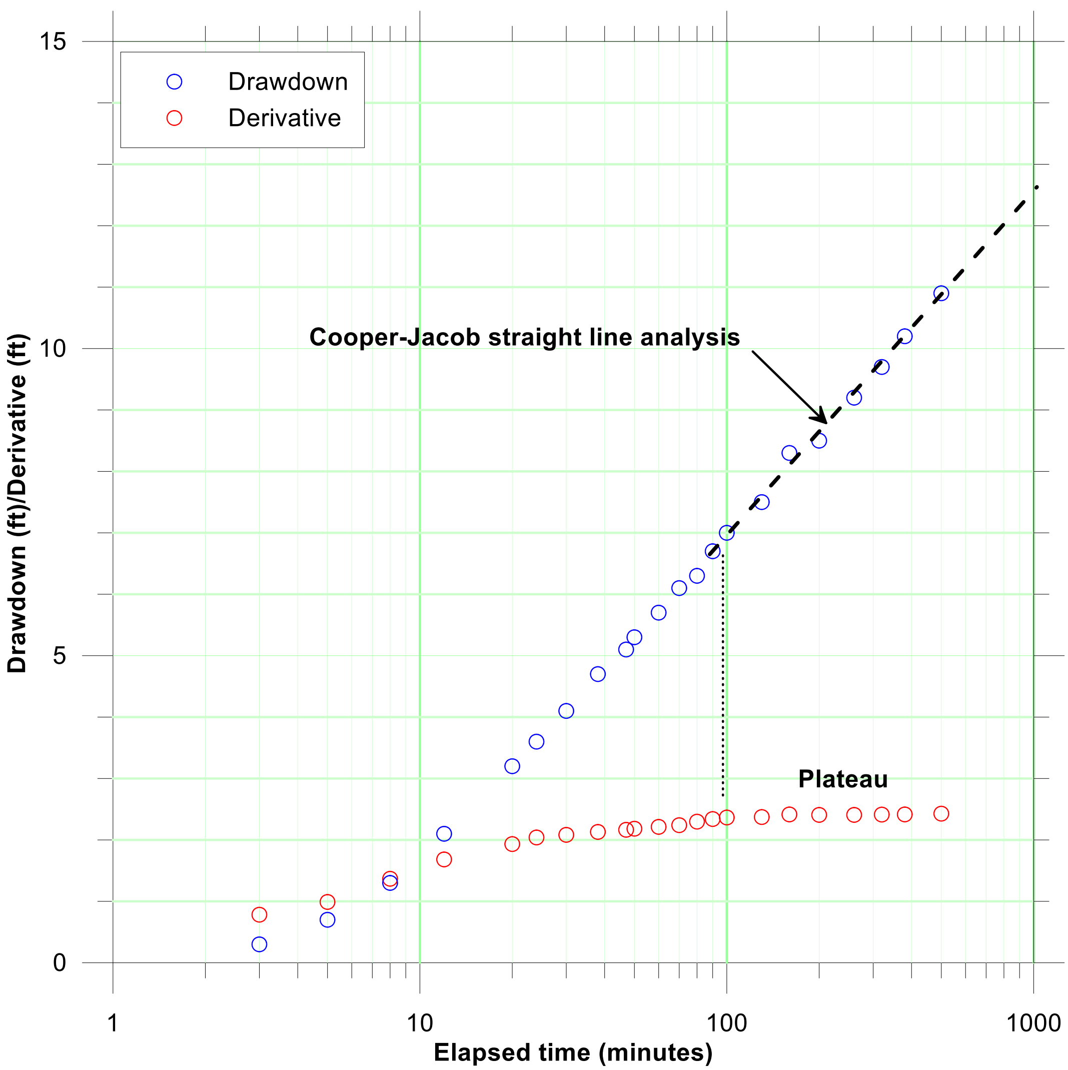


Figure 20. Gridley pumping test, semilog drawdown and derivative plots

10. Cooper-Jacob distance-drawdown analysis

Cooper-Jacob distance-drawdown analyses are not conducted as frequently as time-drawdown analyses; however, they are straightforward to apply and, in many cases, yield representative estimates of transmissivity. The analyses can be helpful in inferring that there are zones with properties that are different with respect to the bulk of the formation and in supporting the detection of boundaries.

The Cooper-Jacob distance-drawdown analysis is developed by differentiating the Cooper-Jacob approximation with respect to $\log_{10}\{r\}$ at a fixed elapsed time, *t*:

$$\left.\frac{\partial s}{\partial[\log\{r\}]}\right|_t = -\,2.303\,\frac{Q}{2\pi T}$$

Defining the *SLOPE*:

$$SLOPE = \frac{\partial s}{\partial(log\, r)} = \Delta s/\log \text{cycle}\, r$$

the transmissivity, *T*, is estimated as:

$$T = 2.303\,\frac{Q}{2\pi}\,\frac{1}{\Delta s}$$

The storativity is estimated by projecting the semilog straight line back to zero drawdown. The intercept along the *r* axis is denoted r_0.

$$s = 0.0 = \frac{Q}{2\pi T}2.303\, log\left[2.2459\frac{Tt}{r_0^2 S}\right]$$

This reduces to:

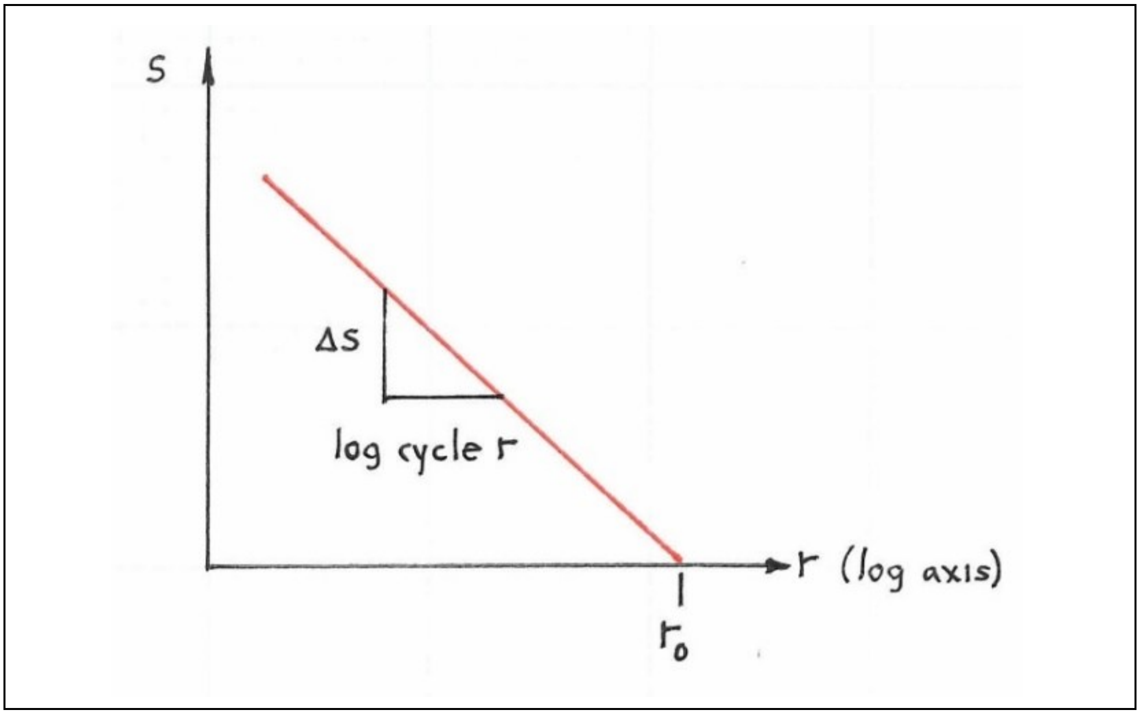
$$\left[2.2459\frac{Tt}{r_0^2 S}\right] = 1.0$$

Solving for *S*:

$$S = 2.2459\,\frac{Tt}{r_0^2}$$

Steps in the Cooper-Jacob distance drawdown analysis

1. For a well at a fixed elapsed time t, plot s vs. r on semilog axes.



2. Estimate the slope, Δs, from the portion of the data that approximates a straight line (drawdown per log cycle r).

3. Estimate the intercept, r_0, by projecting the straight line back to zero drawdown.

4. Estimate the transmissivity and storativity.

$$T = 2.303 \frac{Q}{2\pi} \frac{1}{\Delta s}$$

$$S = 2.2459 \frac{Tt}{r_0^2}$$

5. Assess whether the estimates of T and S and physically realistic.

Example analysis

Wenzel (1936) reported the results from a pumping test conducted in the Platte River Valley near Grand Island, Nebraska. This was an extraordinarily well instrumented test involving 81 observation wells. A map showing the locations of the wells is reproduced here in Figure 21 (Jacob, 1963; Figure 74).

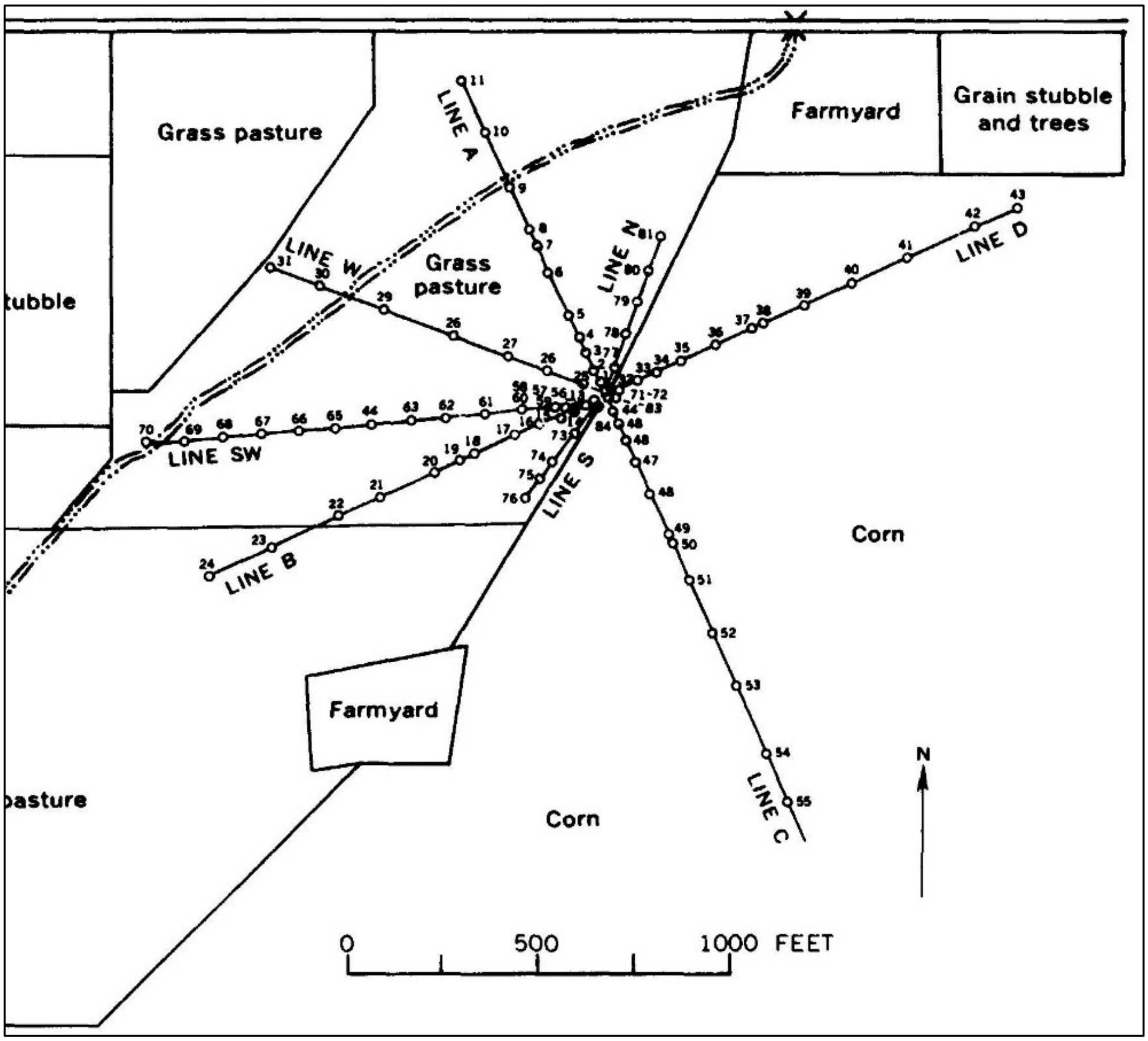


Figure 21. Locations for wells for the pumping test near Grand Island, Nebraska
Reproduced from Jacob (1963)

The pumping test was conducted at a continuous rate of 540 gpm. Wenzel (1936) analyzed the drawdowns after 48 hours using a steady-state approach, developing separate estimates of the transmissivity for seven of the eight lines of wells. The drawdowns for the wells along all of the lines after 48 hours of pumping are plotted in Figure 22.

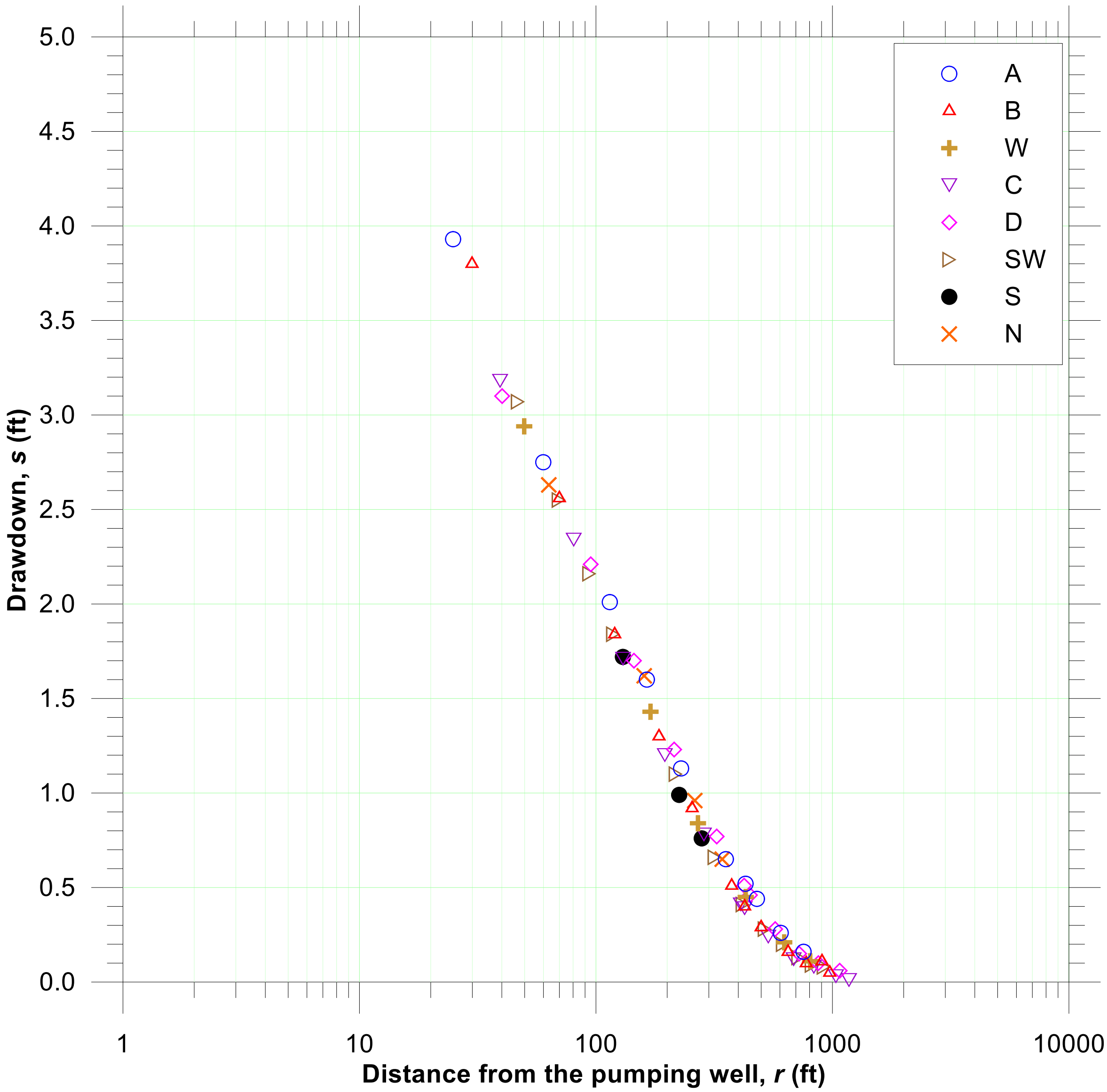


Figure 22. Drawdowns after 48 hours of pumping
Data from Jacob (1963; Table 2)

For distances less than about 300 ft, the drawdowns along all of the lines approximate a single straight line on the semilog plot. Fitting a straight line through this portion of the data yields a transmissivity of 10,500 gpd/ft:

$$T = -2.303\ \frac{(540\ \text{gpm})}{2\pi}(-\ 3.054\ \text{ft})^{-1}\left|\frac{1440\ \text{min}}{\text{d}}\right| = \mathbf{93,300}\ \frac{\mathbf{gpd}}{\mathbf{ft}}$$

Jacob (1963; Table 3) lists seven values of transmissivity that range from 90,000 to 102,000 gpd/ft.

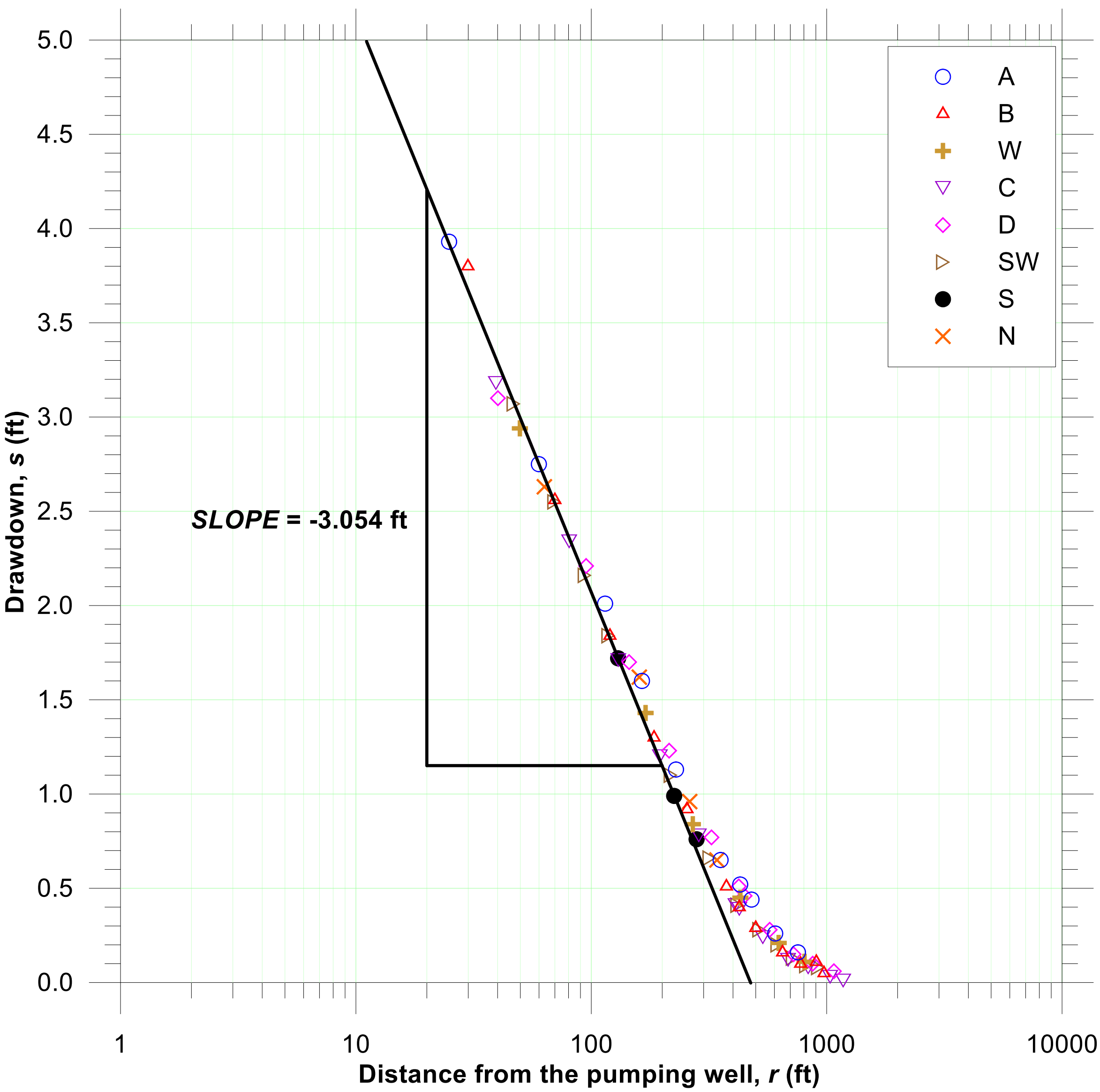


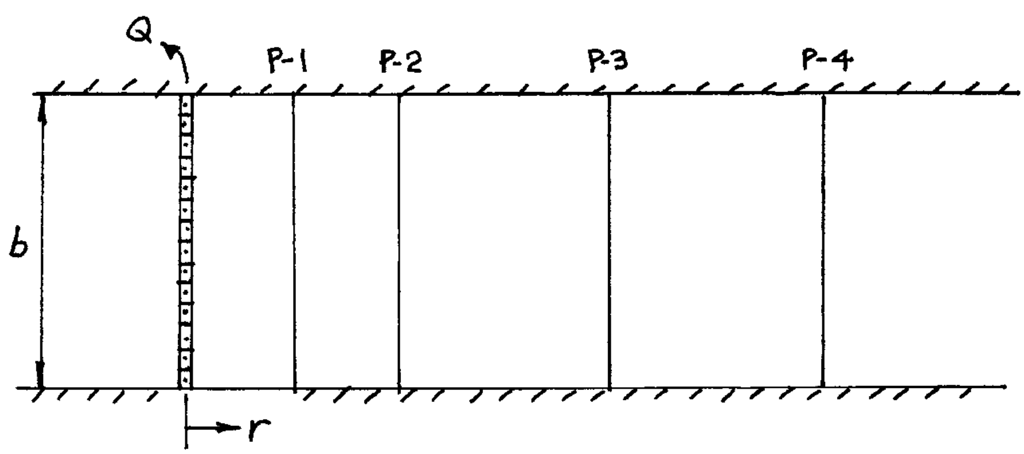
Figure 23 Cooper-Jacob distance-drawdown analysis for Grand Island pumping test

11. Composite analyses

It is important to note that the argument for the Theis solution, u, is expressed in terms of the ratio t/r^2, where t is the elapsed time and r is the distance between an observation well and the pumping well. The solution therefore predicts that the drawdowns for all observation wells completed in the same homogenous aquifer should fall on same curve if they are plotted on an axis of t/r^2.

Example:

A simple example is considered to illustrate this important point. The conceptual model for the example is illustrated in Figure 24. A fully penetrating well is pumped at a constant rate, and the drawdown is monitored at four observation wells.



DATA :

b	:	10.0 m
K_H	:	1×10^{-5} m/s
S_s	:	1×10^{-5} /m
Q	:	5×10^{-4} m^3/s

OBSERVATION POINT	r (m)
P-1	5.0
P-2	10.0
P-3	20.0
P-4	30.0

Figure 24. Example pumping test with multiple observation wells

The time-drawdown records for the individual wells are plotted in Figure 25.

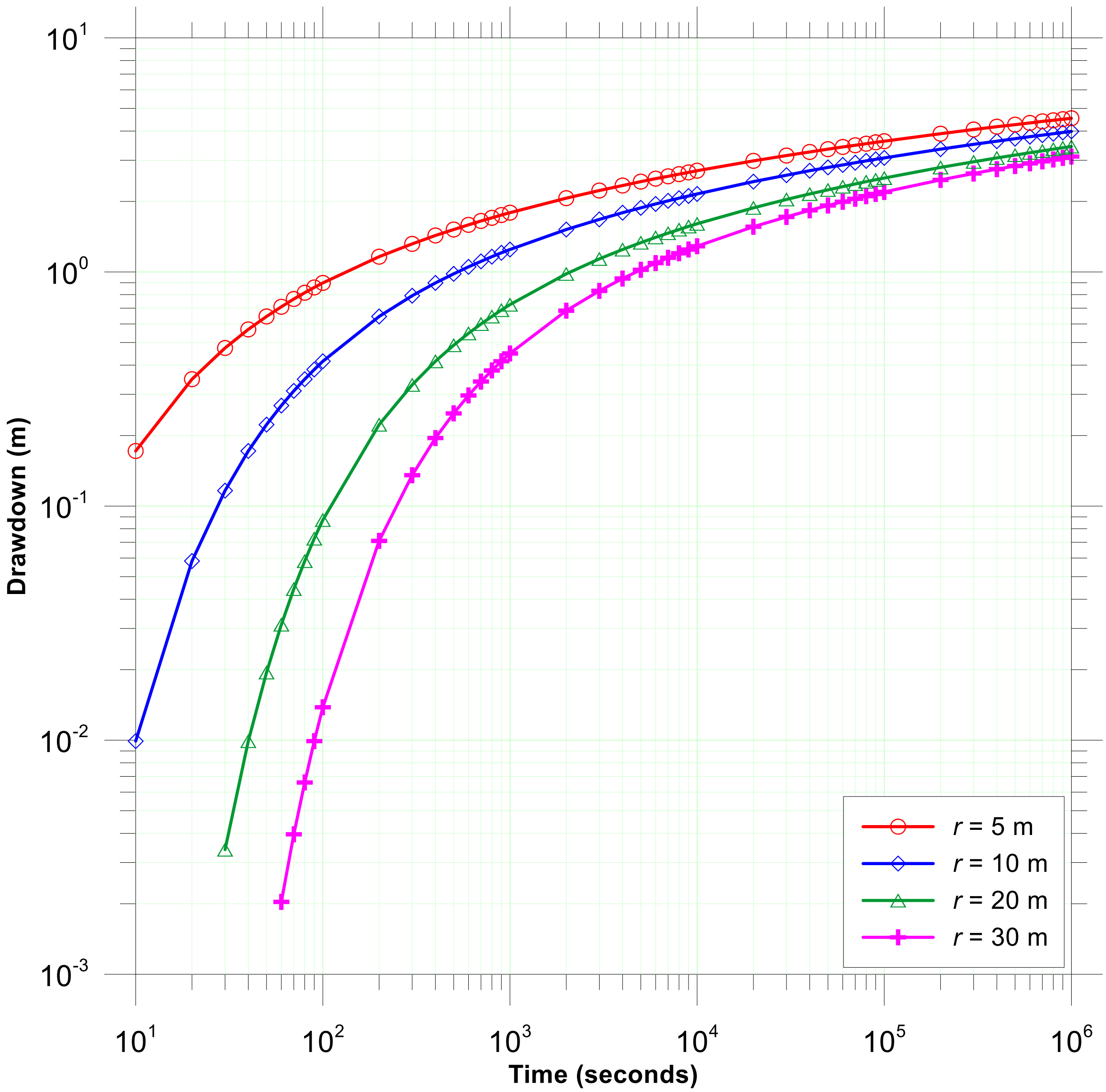


Figure 25. Calculated drawdowns at observation wells

Let us re-plot the data, this time using t/r^2 instead of t to scale elapsed time with respect to the square of the distances between the pumping well and each observation well. Cooper and Jacob (1946) referred to a plot of the drawdowns against t/r^2 as a *composite plot*. As predicted by the Theis solution, the individual drawdown records collapse to a single curve on a composite plot.

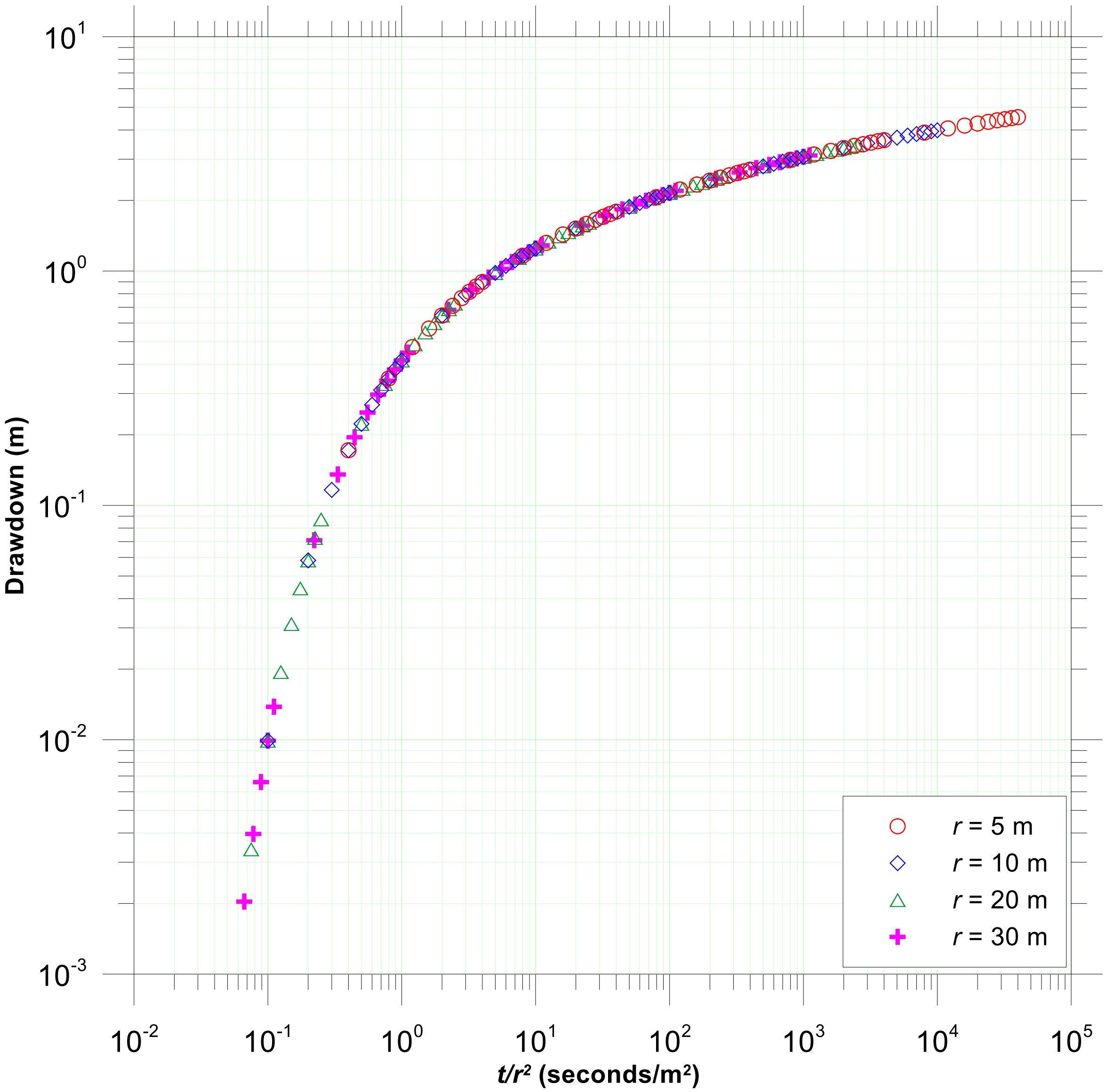


Figure 26. Drawdown data plotted vs. t/r^2

The time-drawdown records for the individual wells are re-plotted on semilog axes in Figure 27.

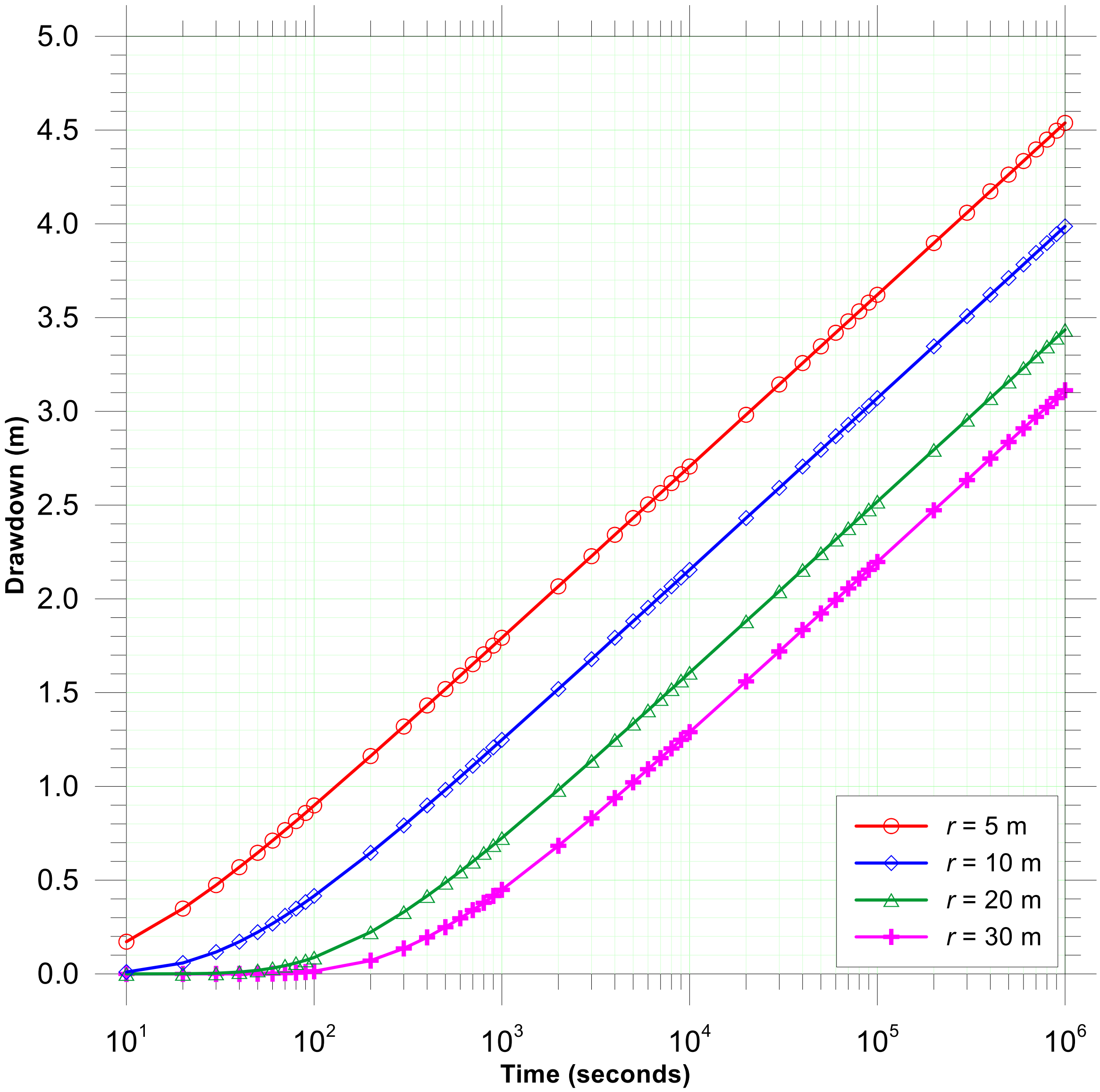


Figure 27. Calculated drawdowns at observation wells, semilog plot

The Cooper-Jacob approximation can be re-written as:

$$s = \frac{Q}{4\pi T} 2.303\, log_{10}\left[2.2459 \frac{T}{S}\left(\frac{t}{r^2}\right)\right]$$

In this form, we see that when the approximation is valid, the drawdown is a linear function of the logarithm of t/r^2. The composite Cooper-Jacob semilog plot for the example is shown in Figure 28. Beyond the limit of applicability of the Cooper-Jacob approximation, the drawdowns collapse to a single straight line.

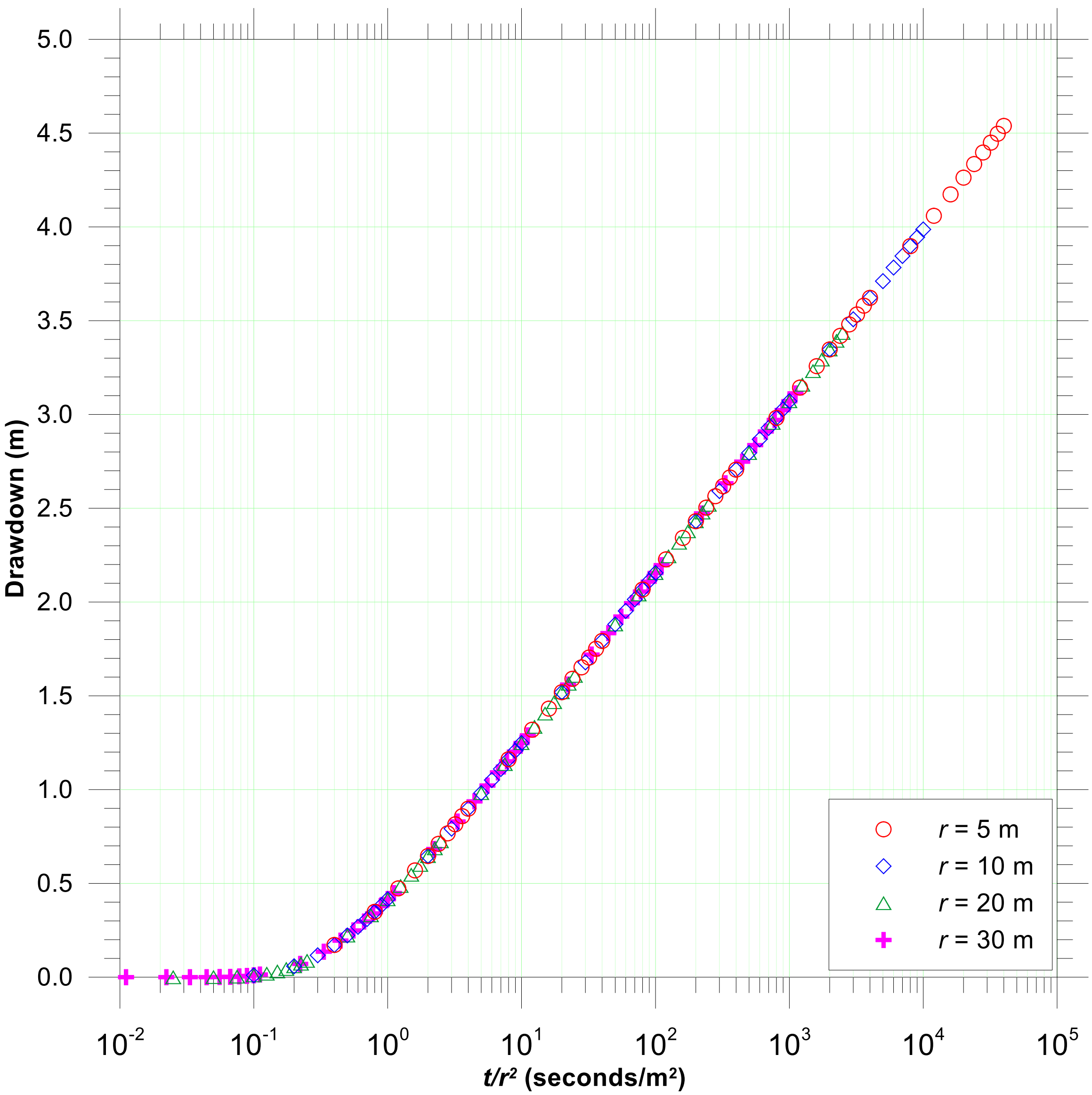


Figure 28. Drawdown data plotted vs. *t/r*²

Cooper-Jacob straight-line analysis on a composite plot

The Cooper-Jacob analysis for a composite plot is essentially identical to the time-drawdown analysis for a single well. The analysis is developed by differentiating the Cooper-Jacob approximation with respect to $log\left(\frac{t}{r^2}\right)$:

$$\frac{\partial s}{\partial\left(log\frac{t}{r^2}\right)} = 2.303\frac{Q}{4\pi T}$$

Defining the *SLOPE*:

$$SLOPE = \frac{\partial s}{\partial\left(log\frac{t}{r^2}\right)} = \Delta s/\text{ log cycle}\frac{t}{r^2}$$

the transmissivity, *T*, is estimated as:

$$T = 2.303\ \frac{Q}{4\pi}\ \frac{1}{\Delta s}$$

The storativity is estimated by projecting the semilog straight line back to zero drawdown. The intercept along the (t/r^2) axis is denoted $(t/r^2)_0$.

$$s = 0.0 = \frac{Q}{4\pi T}2.303\ log_{10}\left[2.2459\frac{T}{S}\left(\frac{t}{r^2}\right)_0\right]$$

This reduces to:

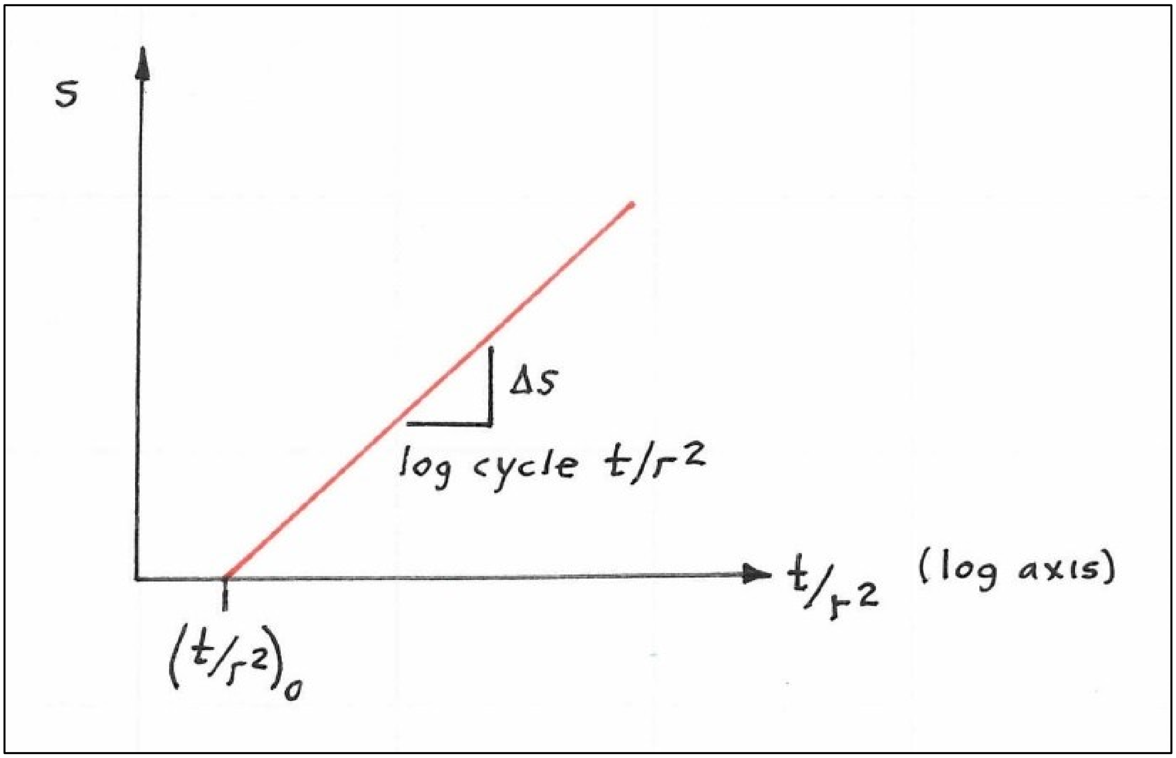
$$\left[2.2459\frac{T}{S}\left(\frac{t}{r^2}\right)_0\right] = 1.0$$

Solving for *S*:

$$S = 2.2459\,T\ \left(\frac{t}{r^2}\right)_0$$

Steps in the Cooper-Jacob straight-line analysis on a composite plot

1. Plot s vs. t/r^2 on semilog axes:



2. Estimate the slope, Δs, from the portion of the data that approximates a straight line (drawdown per log cycle t/r^2).

3. Estimate the intercept, $(t/r^2)_0$, by projecting the straight line back to zero drawdown.

4. Estimate the transmissivity and storativity.

$$T = 2.303 \frac{Q}{4\pi} \frac{1}{\Delta s}$$

$$S = 2.2459\, T \left(\frac{t}{r^2}\right)_0$$

5. Assess whether the estimates of T and S and physically realistic.

The Cooper-Jacob analysis for the example of Figure 24 is shown in Figure 29. As expected, the slope of the straight line yields the transmissivity that was specified to create example calculations.

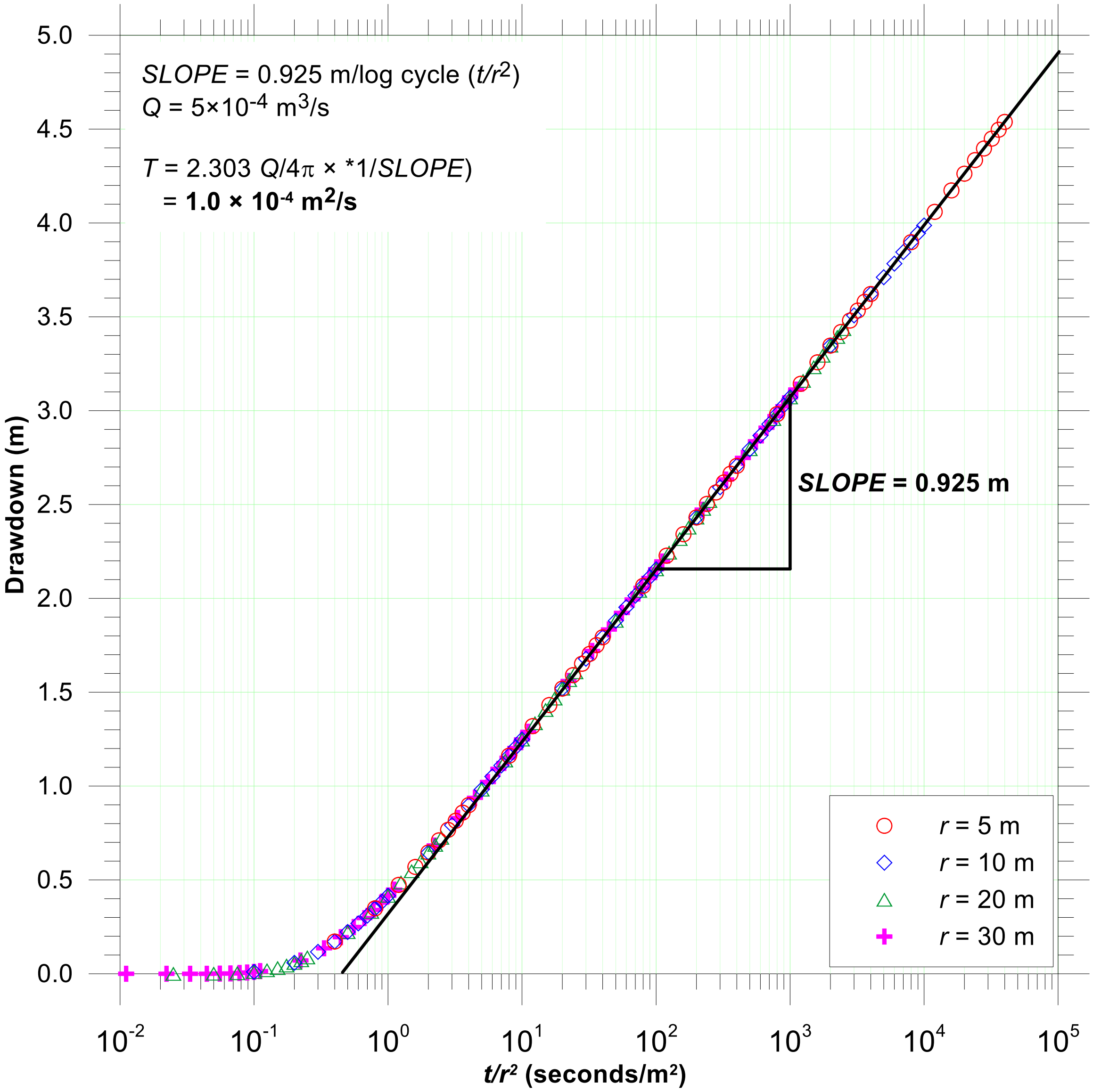


Figure 29. Cooper-Jacob composite analysis

12. Case study: Application of the Cooper-Jacob composite analysis

Schad and Teutsch (1994) reported the results of pumping tests conducted at a well-instrumented site near Stuttgart, Germany. The published data are used to demonstrate the application of the composite analysis. A site map is reproduced in Figure 30.

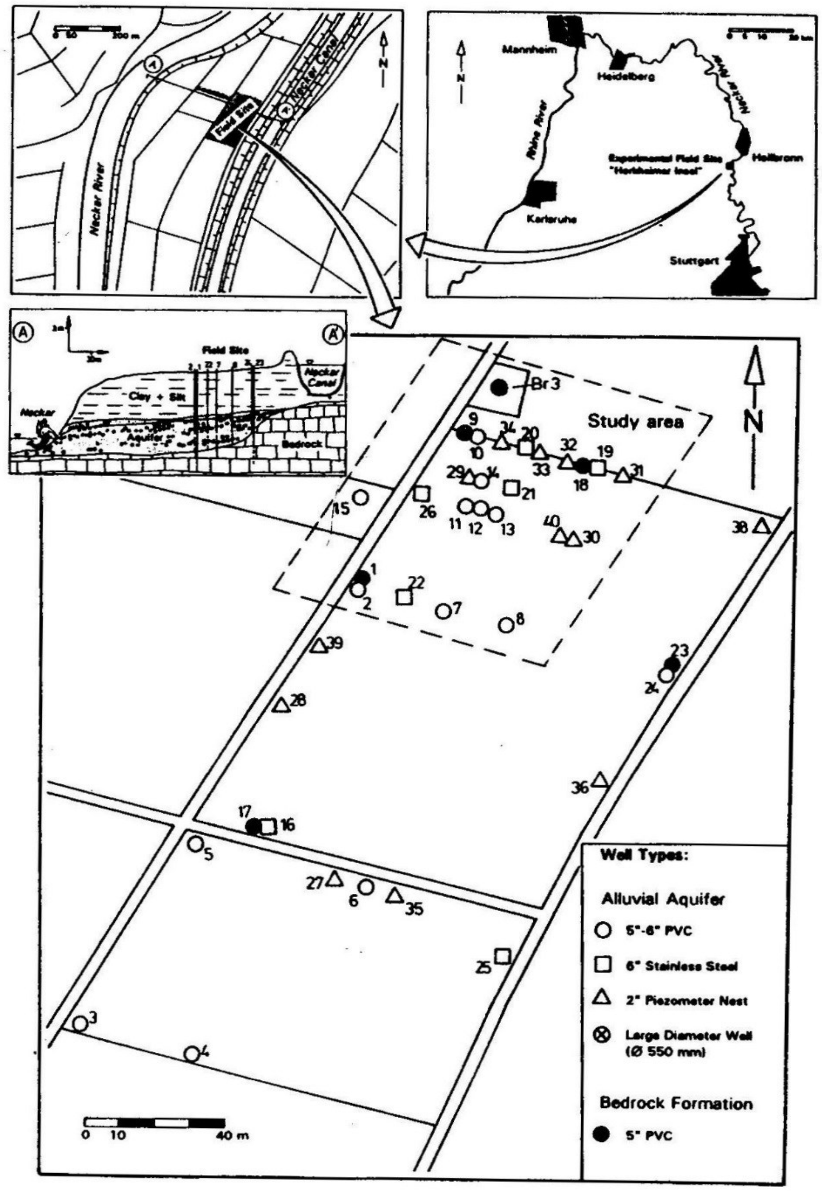


Figure 30. Site map for Horkheimer pumping test

Well Br3 was pumped continuously at an average rate of 13.6 L/sec for four days. The drawdown records for the four observation wells reported in Schad and Teutsch (1994) are reproduced in Figure 31.

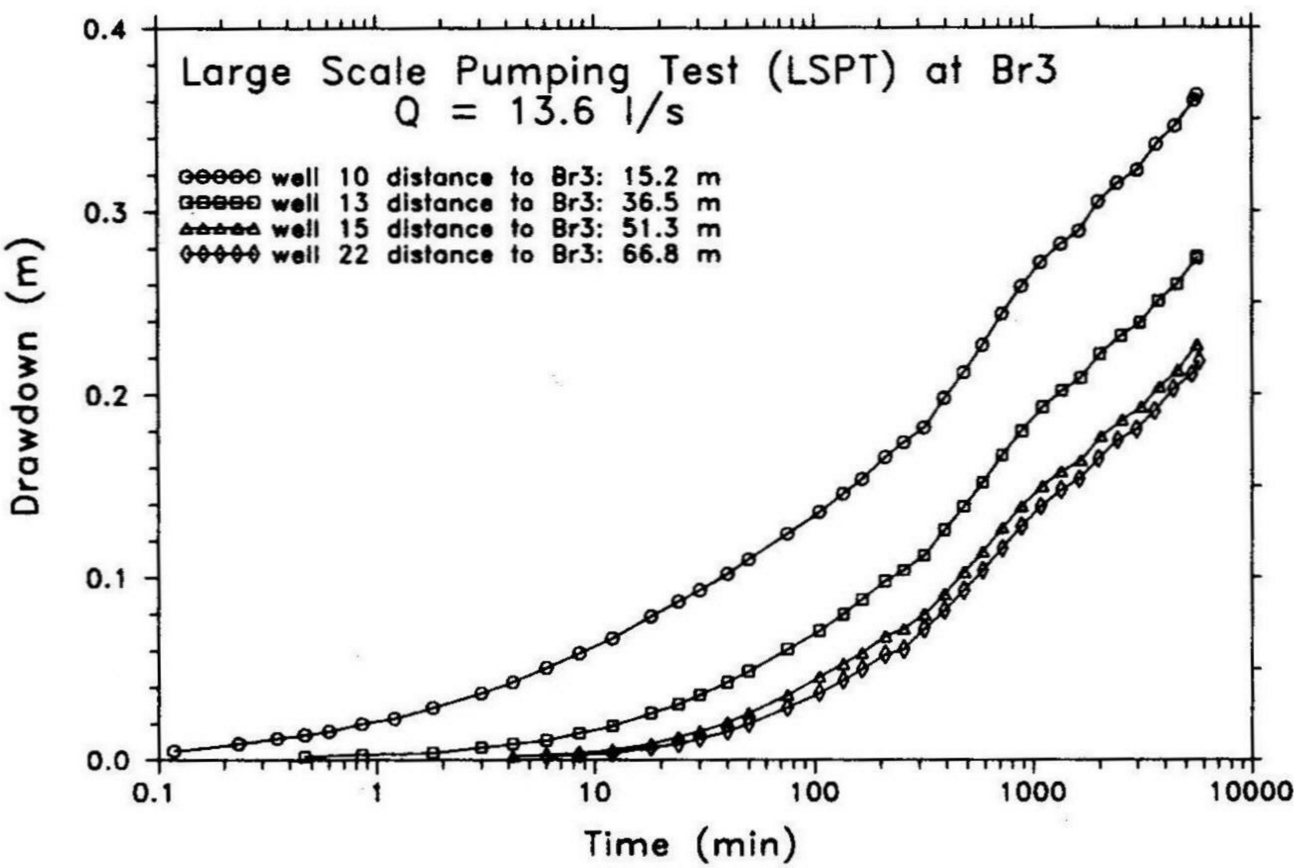


Figure 31. Drawdown data for selected observation wells
Reproduced from Schad and Teutsch (1994)

Original analysis

The reporting of the pumping tests analyses in Schad and Teutsch (1994) are reproduced below. Schad and Teutsch (1994) analyzed the drawdown records for the individual observation wells separately, with three different transmissivity estimates derived from different portions of the responses. Phase one corresponds to "early" time, phase two corresponds to "intermediate" time, and phase three corresponds to "late" time.

Schad and Teutsch (1994) compiled summary statistics of the results of the individual analyses. The reporting in their paper is reproduced below. As shown on the table, the transmissivity estimates from the 0.029 to 0.13 m^2/s, a factor of about 5.

Parameter	LSPT			
Number of tests performed	1			
Number of evaulated drawdown curves	15			
	Min	Mean	Max	CV^c
Radial distances PW[a] – OW[b]	15.2	40.6	70.6	
Transmissivity for phase two ($m^2 s^{-1}$)	0.042	0.065	0.13	0.35
Transmissivity for phase three ($m^2 s^{-1}$)	0.029	0.032	0.035	0.069
Storativity for phase two (-)	0.018	0.035	0.058	0.34
Storativity for phase three (-)	0.026	0.05	0.11	0.37

[a] PW is the pumping well.
[b] OW is the observation well.
[c] *CV* is the coefficient of variation (standard deviation/mean).

Alternative analysis

The drawdown data shown in Figure 31 are re-plotted on a composite plot in Figure 32. After some early-time curvature, the drawdown data from all four observation wells appear to approximate closely a straight line. The single straight-line analysis yields a consistent transmissivity estimate of **0.02 m²/s**. This estimate is less than smallest value reported in Schad and Teutsch (1994).

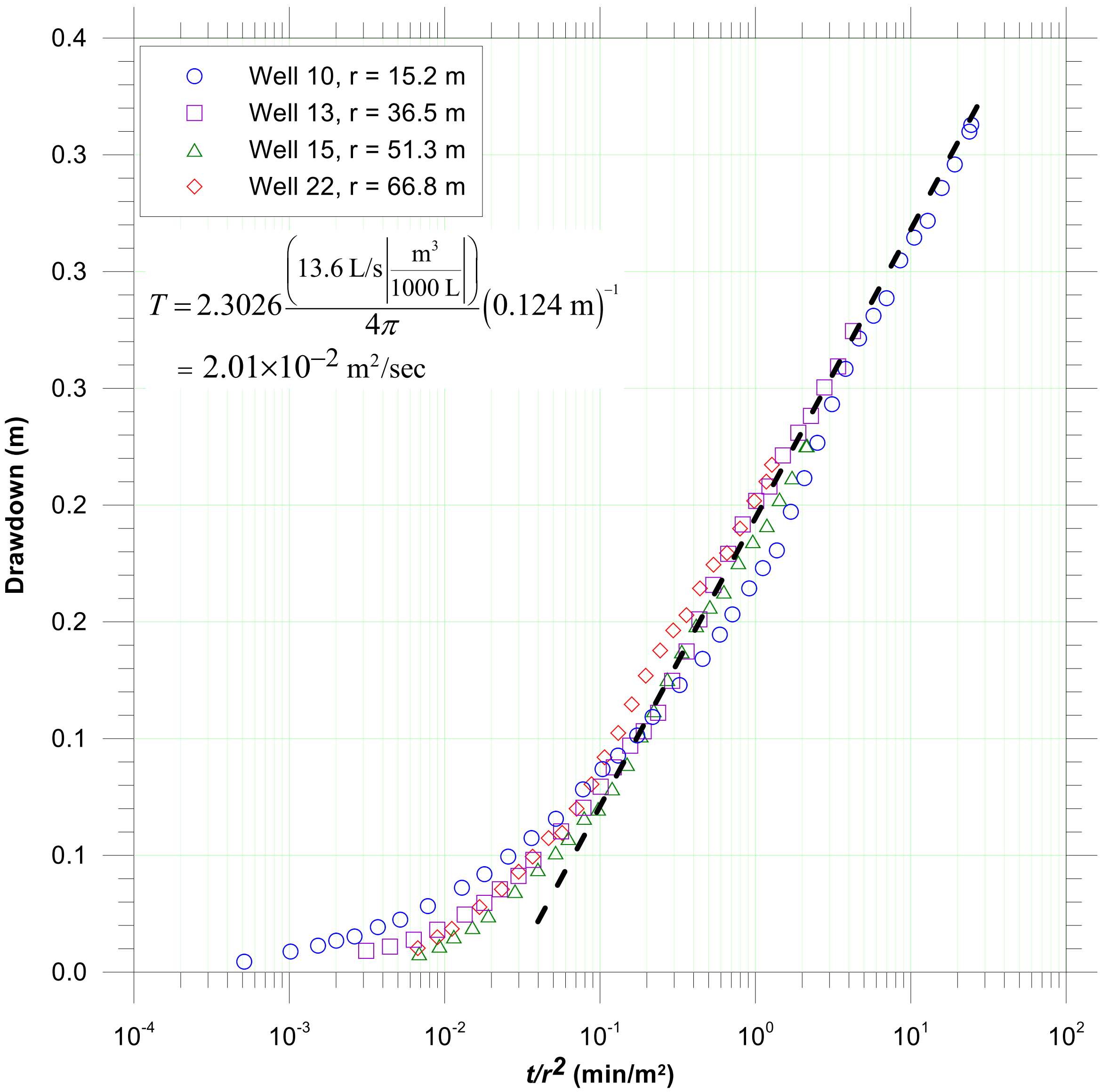


Figure 32. Composite analysis

13. Summary of key points

1. The Theis model provides a benchmark against which the observed responses to pumping at a particular site can be assessed. Checking site conditions against a list of the ideal assumptions allows analysts to identify the conceptual model that best describes their site.

2. The Cooper-Jacob method is the simplest method of interpreting pumping tests in the hydrogeologist's toolkit. This simplicity can be deceptive: the method frequently yields the most reliable estimates of transmissivity. There seems to be little appreciation of its underlying strengths.

3. The conceptual models that underlie the Theis and Cooper-Jacob analyses are identical. The two analyses do not provide independent transmissivity estimates. Rather, the two analyses are complementary. Therefore, it does not really make sense to report separate transmissivity estimates derived from Theis and Cooper-Jacob analyses of the same data. However, there is no reason why an analyst should not employ both methods for the same set of data. Obtaining similar results with both methods confirm that the interpretation is at least internally consistent. It is up to analyst to identify and only report the more reliable transmissivity estimate.

4. The addition of the derivative to the semilog plot increases the defensibility of a Cooper-Jacob straight-line (CJSL) analysis. The simultaneous plotting of the drawdown and the derivative confirms that the derivative reaches a plateau, and helps identify in a direct visual manner the appropriate portion of the response for the analysis.

5. The composite plotting approach is a straightforward extension of the Cooper-Jacob time-drawdown analysis. However, it simplifies the estimation of the representative bulk-average transmissivity and has important diagnostic value. Hydrogeologists should always prepare composite plots when drawdown records are available for more than one observation well.

14. References

Abramowitz, M., and I.A. Stegun, 1972: **Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables**, National Bureau of Standards Applied Mathematics Series 55, U.S. Department of Commerce, U.S. Government Printing Office, Washington DC.

Bourdet, D., T.M. Whittle, A.A. Douglas, and Y.M. Pirard, 1983: A new set of type curves simplifies well test analysis, *World Oil*, vol. 196, no. 6, pp. 95-106.

Bourdet, D., J.A. Ayoub, and Y.M. Pirard, 1989: Use of pressure derivative in well-test interpretation, *SPE Formation Evaluation*, vol. 4, no. 2, pp. 293-302.

Carslaw, H.S., 1921: **Introduction to the Mathematical Theory of the Conduction of Heat in Solids**, MacMillan and Co., Ltd., London, UK.

Cooper, H.H., Jr., and C.E. Jacob, 1946: A generalized graphical method for evaluating formation constants and summarizing well-field history, *Transactions of the American Geophysical Union*, vol. 27, no. 4, pp. 526-534.

Jacob, C.E., 1963: Determining the permeability of water-table aquifers, in *Methods of Determining Permeability, Transmissibility and Drawdown*, compiled by R. Bentall, U.S. Geological Survey, Water-Supply Paper 1536-I, pp. 245-271.

Matthews, C.S., and D.G. Russell, 1967: **Pressure Buildup and Flow Tests in Wells**, Society of Petroleum Engineers of AIME, New York, New York.

Schad, H., and G. Teutsch, 1994: Effects of the investigation scale on pumping test results in heterogeneous porous aquifers, *Journal of Hydrology*, Vol. 159, pp. 61-77.

Theis, C.V., 1935: The relation between the lowering of the piezometric surface and the rate and duration of discharge of a well using ground-water storage, *Transactions of the American Geophysical Union*, 16th Annual Meeting, Part 2, pp. 519-524.

Todd, D.K., and L.W. Mays, 2005: **Groundwater Hydrology**, 3rd edition, John Wiley & Sons, Inc., Hoboken, New Jersey.

Walton, W.C., 1970: **Groundwater Resource Evaluation**, McGraw-Hill, New York, New York.

Wenzel, L.K., 1936: The Thiem method for determining permeability of water-bearing materials and its application to the determination of specific yield: Results of investigations in the Platte River Valley, Nebr., U.S. Geological Survey, Water-Supply Paper 887, 192 p.

THE RELATION BETWEEN THE LOWERING OF THE PIEZOMETRIC SURFACE AND THE RATE AND DURATION OF DISCHARGE OF A WELL USING GROUND-WATER STORAGE

Charles V. Theis

When a well is pumped or otherwise discharged, water-levels in its neighborhood are lowered. Unless this lowering occurs instantaneously it represents a loss of storage, either by the unwatering of a portion of the previously saturated sediments if the aquifer is nonartesian or by release of stored water by the compaction of the aquifer due to the lowered pressure if the aquifer is artesian. The mathematical theory of ground-water hydraulics has been based, apparently entirely, on a postulate that equilibrium has been attained and therefore that water-levels are no longer falling. In a great number of hydrologic problems, involving a well or pumping district near or in which water-levels are falling, the current theory is therefore not strictly applicable. This paper investigates in part the nature and consequences of a mathematical theory that considers the motion of ground-water before equilibrium is reached and, as a consequence, involves time as a variable.

To the extent that Darcy's law governs the motion of ground-water under natural conditions and under the artificial conditions set up by pumping, an analogy exists between the hydrologic conditions in an aquifer and the thermal conditions in a similar thermal system. Darcy's law is analogous to the law of the flow of heat by conduction, hydraulic pressure being analogous to temperature, pressure-gradient to thermal gradient, permeability to thermal conductivity, and specific yield to specific heat. Therefore, the mathematical theory of heat-conduction developed by Fourier and subsequent writers is largely applicable to hydraulic theory. This analogy has been recognized, at least since the work of Slichter, but apparently no attempt has been made to introduce the function of time into the mathematics of ground-water hydrology. Among the many problems in heat-conduction analogous to those in ground-water hydraulics are those concerning sources and sinks, sources being analogous to recharging wells and sinks to ordinary discharging wells.

C. I. Lubin, of the University of Cincinnati, has with great kindness prepared for me the following derivation of the equation expressing changes in temperature due to the type of source or sink that is analogous to a recharging or discharging well under certain ideal conditions, be discussed below.

The equation given by H. S. Carslaw (Introduction to the mathematical theory of the conduction of heat in solids, 2nd ed., p. 152, 1921) for the temperature at any point in an infinite plane with initial temperature zero at any time due to an "instantaneous line-source coinciding with the axis of z of strength Q" (involving two-dimensional flow of heat) is

$$v = (Q/4\pi\kappa t)\, e^{-(x^2 + y^2)/4\kappa t} \tag{1}$$

where v = change in temperature at the point x,y at the time t; Q = the strength of the source or sink--in other words, the amount of heat added or taken out instantaneously divided by the specific heat per unit-volume; κ = Kelvin's coefficient of diffusivity, which is equal to the coefficient of conductivity divided by the specific heat per unit-volume; and t = time.

The effect of a continuous source or sink of constant strength is derived from equation (1) as follows: Let $Q = \varphi(t')dt'$; then $v_{(x,y,t)} = \int_0^t [\varphi(t')/4\pi\kappa(t-t')]\, e^{-(x^2+y^2)/4\kappa(t-t')}\, dt'$. Let $\varphi(t') = \lambda$, a constant; then $v_{(t)} = (\lambda/4\pi\kappa)\int_0^t [e^{-(x^2+y^2)/4\kappa(t-t')}/(t-t')]\, dt'$. Let $u = (x^2+y^2)/4\kappa(t-t')$; then

$$v_{(t)} = (\lambda/4\pi\kappa)\int_{(x^2+y^2)/4\kappa t}^{\infty} [e^{-u}/(t-t')]\, [(x^2+y^2)/4\kappa]\, [du/u^2]$$

$$= (\lambda/4\pi\kappa)\int_{(x^2+y^2)/4\kappa t}^{\infty} (e^{-u}/u)du \tag{2}$$

The definite integral, $\int_{(x^2+y^2)/4\kappa t}^{\infty} (e^{-u}/u)du$, is a form of the exponential integral, tables of which are available (Smithsonian Physical Tables, 8th rev. ed., table 32, 1933; the values to be used are those given for Ei (-x), with the sign changed). The value of the integral is given by the series

$$\int_x^{\infty} (e^{-u}/u)du = -0.577216 - \log_e x + x - x^2/2\cdot 2! + x^3/3\cdot 3! - x^4/4\cdot 4! + \ldots\ldots \tag{3}$$

Equation (2) can be immediately adapted to ground-water hydraulics to express the draw-down at any point at any time due to pumping a well. The coefficient of diffusivity, , is analogous to the coefficient of transmissibility of the aquifer divided by the specific yield. (The term "coefficient of transmissibility" is here used to denote the product of Meinzer's coefficient of permeability and the thickness of the saturated portion of the aquifer; it quantitatively describes the ability of the aquifer to transmit water. Meinzer's coefficient of permeability denotes a characteristic of the material; the coefficient of transmissibility denotes the analogous characteristic of the aquifer as a whole.) The continuous strength of the sink is analogous to the pumping rate divided by the specific yield. Making these substitutions, we have

$$v = (F/4\pi T)\int_{r^2s/4Tt}^{\infty} (e^{-u}/u)du \tag{4}$$

in which the symbols have the meanings given with equation (5). In equation (4) the same units must of course be used throughout. Equation (4) may be adapted to units commonly used

$$v = (114.6F/T)\int_{1.87r^2s/Tt}^{\infty} (e^{-u}/u)du \tag{5}$$

where v = the draw-down, in feet, at any point in the vicinity of a well pumped at a uniform rate; F = the discharge of the well, in gallons a minute; T = the coefficient of transmissibility of aquifer, in gallons a day, through each 1-foot strip extending the height of the aquifer, under a unit-gradient--this is the average coefficient of permeability (Meinzer) multiplied by the thickness of the aquifer; r = the distance from pumped well to point of observation, in feet; s = the specific yield, as a decimal fraction; and t = the time the well has been pumped, in days.

Equation (5) gives the draw-down at any point around a well being pumped uniformly (and continuously) from a homogeneous aquifer of constant thickness and infinite areal extent at any time. The introduction of the function, time, is the unique and valuable feature of the equation. Equation (5) reduces to Thiem's or Slichter's equation for artesian conditions when the time of pumping is large.

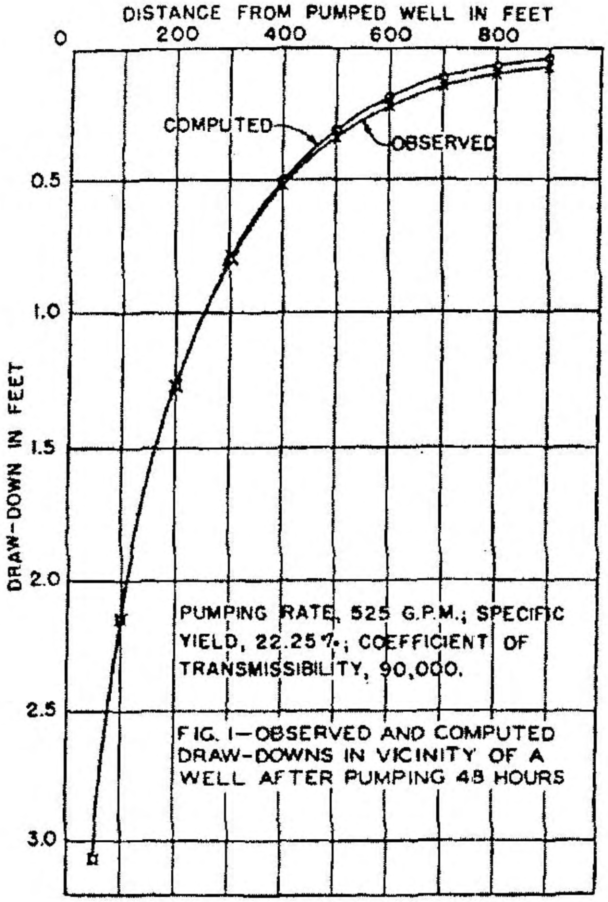


FIG. 1—OBSERVED AND COMPUTED DRAW-DOWNS IN VICINITY OF A WELL AFTER PUMPING 48 HOURS

Empirical tests of the equation are best made with the data obtained by L. K. Wenzel (Recent investigations of Thiem's method for determining permeability of water-bearing sediments. Trans. Amer. Geophys. Union, 13th annual meeting, pp. 313-317, 1932; also Specific yield determined from a Thiem's pumping test, Trans. Amer. Geophys. Union, 14th annual meeting, pp. 475-477, 1933) from pumping tests in the Platte Valley in Nebraska. Figure 1 presents the comparison of the computed and observed draw-downs after two days of pumping. The observed values are those of the generalized depression of the water-table as previously determined by Mr. Wenzel. The computed values are obtained by equation (5), using values of permeability and specific yield that are within one per cent of those determined by Mr. Wenzel by other methods. The agreement represented may be regarded as showing either that the draw-downs have been computed from known values of transmissibility and specific yield or that these factors have been computed from the known draw-downs.

Theoretically, the equation applies rigidly only to water-bodies (1) which are contained in entirely homogeneous sediments, (2) which have infinite areal extent, (3) in which the well penetrates the entire thickness of the water-body, (4) in which the coefficient of transmissibility is constant at all times and in all places, (5) in which the pumped well has an infinitesimal diameter, and (6) - applicable only to unconfined water-bodies - in which the water in the volume of sediments through which the water-table has fallen is discharged instantaneously with the fall of the water-table.

These theoretical restrictions have varying degrees of importance in practice. The effect of heterogeneity in the aquifer can hardly be foretold. The effect of boundaries can be considered by more elaborate analyses, once they are located. The effect of the well failing to penetrate the entire aquifer is apparently negligible in many cases. The pumped well used in the set-up that yielded the data for Figure 1 penetrated only 30 feet into a 90-foot aquifer. The coefficient of transmissibility must decrease during the process of pumping under water-table conditions, because of the diminution in the cross-section of the area of flow due to the fall of the water-table; however, it appears from Figure 1 that if the water-table falls through a distance equal only to a small percentage of the total thickness of the aquifer the errors are not large enough to be observed. In artesian aquifers the coefficient of transmissibility probably decreases because of the compaction of the aquifer, but data on this point are lacking. The error due to the finite diameter of the well is apparently always insignificant.

In heat-conduction a specific amount of heat is lost concomitantly and instantaneously with fall in temperature. It appears probable, analogously, that in elastic artesian aquifers a specific amount of water is discharged instantaneously from storage as the pressure falls. In nonartesian aquifers, however, the water from the sediments through which the water-table has fallen drains comparatively slowly. This time-lag in the discharge of the water made available from storage is neglected in the mathematical treatment here given. Hence an error is always present in the equation when it is applied to water-table conditions. However, inasmuch as the rate of fall of the water-table decreases progressively after a short initial period, it seems probable that as pumping continues the rate of drainage of the sediments tends to catch up with the rate of fall of the water-table, and hence that the error in the equation becomes progressively smaller.

For instance, although the draw-downs computed for a 24-hour period of pumping in Mr. Wenzel's test showed a definite lack of agreement with the observations, similar computations for a

48-hour period gave the excellent agreement shown in Figure 1. Unfortunately data for periods of pumping longer than 48 hours have not been available.

The equation implies that any two observations of draw-down, whether at different places or at the same place at different times, are sufficient to allow the computation of specific yield and transmissibility. However, more observations are always necessary in order to guard against the possibility that the computations will be vitiated by the heterogeneity of the aquifer. Moreover, it appears that the time-lag in the drainage of the unwatered sediments makes it impossible at present to compute transmissibility and specific yield from observations on water-levels in only one observation-well during short periods of pumping. Good data from artesian wells have not been available, but such data as we have hold out the hope that transmissibility and specific yield may be determined from data from only one observation-well.

A useful corollary to equation (5) may be derived from an analysis of the recovery of a pumped well. If a well is pumped for a known period and then left to recover, the residual draw-down at any instant will be the same as if pumping of the well had been continued but a recharge well with the same flow had been introduced at the same point at the instant pumping stopped. The residual draw-down at any instant will then be

$$v' = (114.6F/T)\left[\int_{1.87r^2s/Tt}^{\infty}(e^{-u}/u)du - \int_{1.87r^2s/Tt'}^{\infty}(e^{-u}/u)du\right] \qquad (6)$$

where t is the time since pumping started and t' is the time since pumping stopped.

In and very close to the well the quantity $(1.87r^2s/Tt')$ will be very small as soon as t' ceases to be small, because r is very small. In many problems ordinarily met in ground-water hydraulics, all but the first two terms of the series of equation (3) may be neglected, so that, if $Z = (1.87r^2s/Tt)$ and $Z' = (1.87r^2s/Tt')$ equation (6) may be approximately rewritten

$$v' = (114.6F/T)\,[-0.577 - \log_e Z + 0.577 + \log_e Z(t/t')] = (114.6F/T)\log_e(t/t')$$

Transposing and converting to common logarithms, we have

$$T = (264F/v')\log_{10}(t/t') \qquad (7)$$

This equation permits the computation of the coefficient of transmissibility of an aquifer from an observation of the rate of recovery of a pumped well.

Figure 2 shows a plot of observed recovery-curves. The ordinates are log (t/t'); the abscissas are the distances the water-table lies below its equilibrium-position. According to equation (7) the points should fall on a straight line passing through the origin. Curve A is a plot of the recovery of a well within 3 feet of the well pumped for Mr. Wenzel's test, previously mentioned. Most of the points lie on a straight line, but the line passes below the origin. This discrepancy is probably due to the fact that the water-table rises faster than the surrounding pores are filled. The coefficient of permeability computed from the equation is about 1200, against a probably correct figure of 1000. Curve B is plotted from data obtained from an artesian well near Salt Lake City. The points all fall according to theory.

Curve C shows the recovery of a well penetrating only the upper part of a nonartesian aquifer of comparatively low transmissibility. It departs markedly from a straight line. This curve probably follows equation (6), but it does not follow equation (7), for in this case $(1.87r^2s/Tt')$ is not small. Equation (6), involving r and s, neither of which may be known in practice, is not of practical value for the present purpose. Further empirical tests may show that it is feasible to project the curve to the origin, in the neighborhood of which $(1.87r^2s/Tt')$ becomes small, owing to the increase in t and t', and apply equation (7) to the extrapolated values so obtained in order to determine at least an approximate value of the transmissibility.

The paramount value of equation (5) apparently lies in the fact that it gives part of the theoretical background for predicting the future effects of a given pumping regimen upon the water-levels in a district that is primarily dependent on ground-water storage. Such districts may include many of those tapping extensive nonartesian bodies of ground-water. Figure 3 shows the vertical rate of fall of the water-level in an infinite aquifer, the water being all taken from storage. The curves are plotted for certain definite values of pumping rate, transmissibility, and specific yield, but by changing the scales either curve could be made applicable to any values set up.

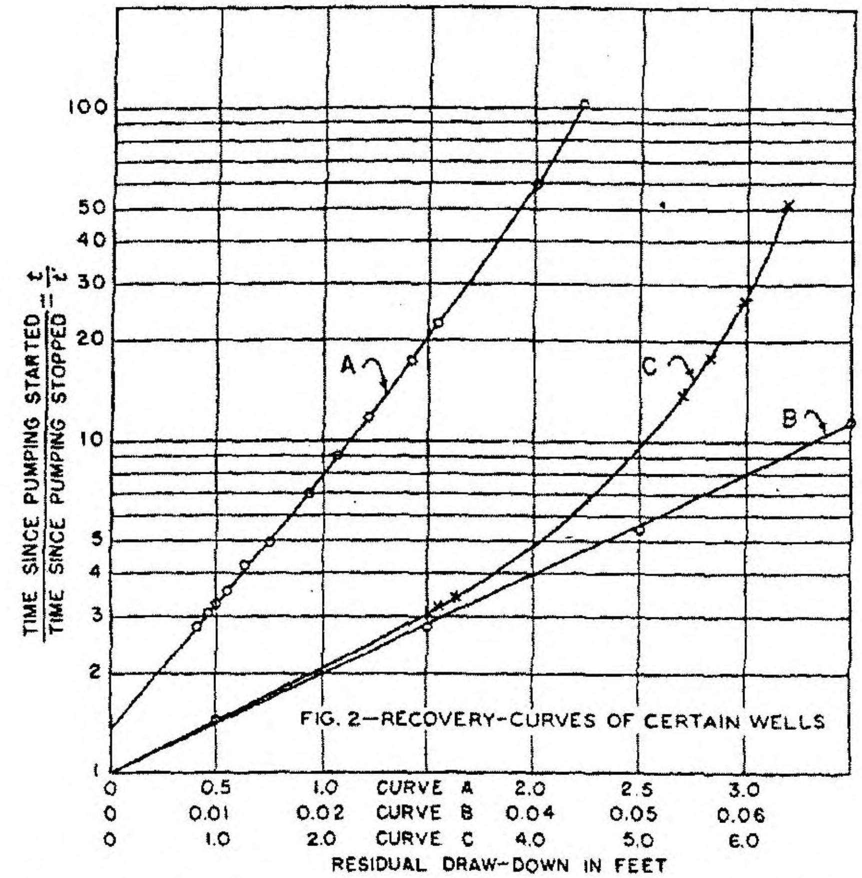


FIG. 2—RECOVERY-CURVES OF CERTAIN WELLS

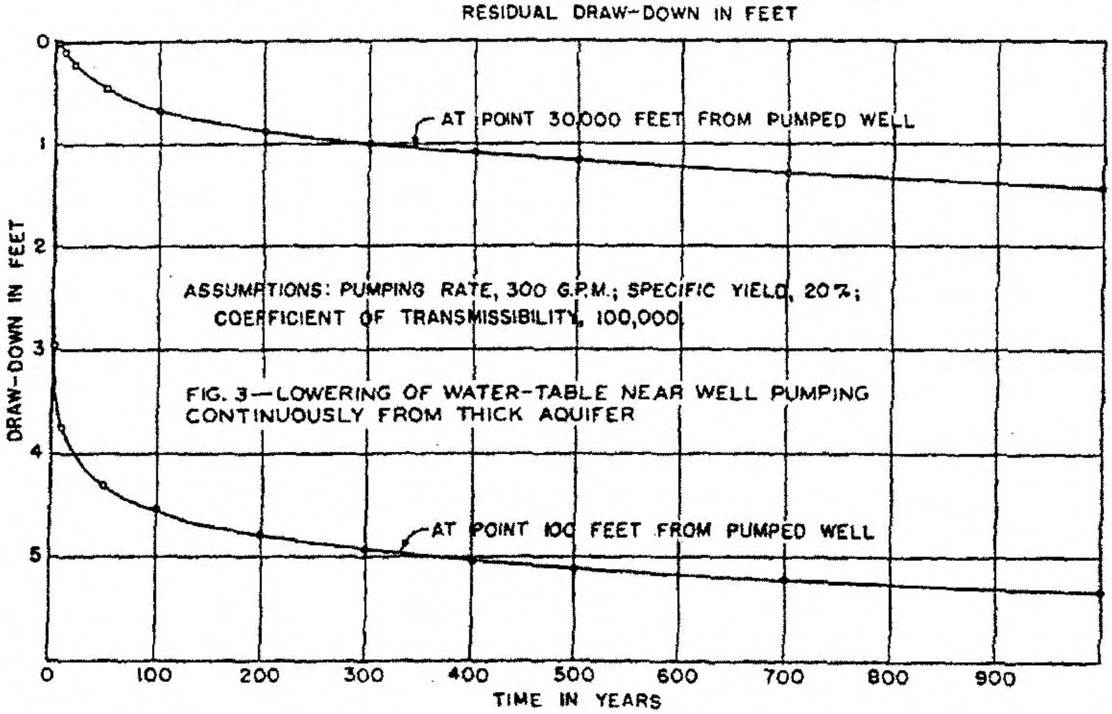


FIG. 3—LOWERING OF WATER-TABLE NEAR WELL PUMPING CONTINUOUSLY FROM THICK AQUIFER

These theoretical curves agree qualitatively with the facts generally observed when a well is pumped. The water-level close to the well at first falls very rapidly, but the rate of fall soon slackens. In the particular case considered in Figure 3 the water-level at a point 100 feet from the pumped well would fall during the first year of pumping more than half the distance it would fall in 1000 years. A delayed effect of the pumping is shown at distant points.

The water-level at a point about 6 miles from the pumped well of Figure 3 would fall only minutely for about five years but would then begin to fall perceptibly, although at a much less rate than the water-level close to the well. Incidentally the rate of fall after considerable pumping is so small that it might easily lead to a false assumption of equilibrium. The danger in a pumping district using ground-water storage lies in the delayed interference of the wells. For instance, although in 50 years one well would cause a draw-down of only 6 inches in another well 6 miles away, yet the 100 wells that might lie within 6 miles of a given well would cause in it a total draw-down of more than 50 feet.

In the preparation of this paper I have had the indispensable help not only of Dr. Lubin, who furnished the mathematical keystone of the paper, but also of Dr. C. E. Van Orstrand, of the United States Geological Survey, and of my colleagues of the Ground Water Division of the Survey, who cordially furnished data and criticism.

U. S. Geological Survey,
Washington, D. C.

CJN 2022/10/04

Theis solution in "contemporary" notation

1. Governing equation

$$S \frac{\partial s}{\partial t} = T\left(\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r}\right) \qquad 0 < r < \infty$$

2. Boundary conditions

$$\lim_{r \to 0} 2\pi T r \frac{\partial s}{\partial r} = -Q$$

$$s(\infty, t) = 0$$

3. Solution

$$s(r,t) = \frac{Q}{4\pi T} \int_u^\infty \frac{1}{y} \text{EXP}\{-y\}\, dy$$

$$u = \frac{r^2 S}{4Tt}$$

4. Theis solution in U.S. common units

$[s] = ft$

$[r] = ft$

$[t] = days$

$[T] = gallons/day$

$[Q] = gallons/minute$

$$u = \frac{r^2 S}{4Tt} \left| \frac{7.4805 \text{ gal}}{ft^3} \right| = 1.87 \frac{r^2 S}{Tt}$$

$$s(r,t) = \frac{Q}{4\pi T} W(u) \left| \frac{1440 \text{ min}}{\text{day}} \right| = 114.59 \frac{Q}{T} W(u)$$

These expressions are consistent with Theis (1935) Equation (5). //

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A GENERALIZED GRAPHICAL METHOD FOR EVALUATING FORMATION CONSTANTS AND SUMMARIZING WELL-FIELD HISTORY

H. H. Cooper, Jr. and C. E. Jacob

(Published with the approval of the Director of the Geological Survey, United States Department of the Interior)

Abstract--The capacities of a water-bearing formation to transmit water under a hydraulic gradient and to yield water from storage when the water table or artesian pressure declines, are generally expressed, respectively, in terms of a coefficient of transmissibility and a coefficient of storage. Determinations of these two constants are almost always involved in quantitative studies of ground-water problems.

C. V. THEIS [1935, see "References" at end of paper] gave an equation, adapted from the solution of the analogous problem in heat conduction, for computing the non-steady drawdown accompanying the radial flow of water to a well of constant discharge. This equation has been used successfully many times for determining coefficients of transmissibility and storage from observed drawdowns. As it involves a transcendental function known as the exponential integral and two unknown coefficients, one of which occurs both in the argument and as a divisor of the function, the coefficients cannot be determined directly. However, they may be determined by a graphical method devised by THEIS and described by JACOB [1940, p. 582] and WENZEL [1942, pp. 88-89]. This method requires the use of a "type curve," on which the observed data are superimposed to determine the coefficients.

Later, WENZEL and GREENLEE [1944] gave a generalization of THEIS' graphical method by which the coefficients may be determined from tests of one or more discharging wells operated at changing rates. This method requires the computation of a special type curve for each observation of drawdown used. It is without doubt a worth-while contribution to the quantitative techniques of ground-water hydraulics, but in tests that involve more than a very few discharging wells or a very few changes in the rates of discharge, the computation of the special type curves is necessarily so laborious as to make the method difficult to apply.

The present paper gives a simple straight-line graphical method for accomplishing the same purposes as the methods developed by THEIS and by WENZEL and GREENLEE. Type curves are not required. The writers believe that the straight-line method, where applicable, has decided advantages, in ease of application and interpretation, over the other graphical methods. However, as the method will not be applicable in some cases, it is expected to supplement, rather than supersede, the other methods. The method is designed especially for artesian conditions, but it may be applied successfully to tests of non-artesian aquifers under favorable circumstances.

This paper first gives the development of the method for tests involving a single discharging well operating at a steady rate, and then generalizes the method to make it applicable to tests involving one or more wells discharging intermittently or at changing rates. Examples are given to demonstrate the method.

Straight-line method for a single well discharging at a steady rate

When sufficient time has elapsed after an artesian well has begun discharging at a steady rate, the drawdown within a given distance increases approximately in proportion to the logarithm of the time since the discharge began, and decreases in proportion to the logarithm of the distance from the well. By virtue of this relationship, it is possible to determine the coefficients of transmissibility and storage of an aquifer from a simple semi-logarithmic plot of observed drawdowns.

The drawdown produced by a well discharging at a steady rate from an extensive artesian aquifer of uniform thickness and permeability is given by equation (1) [THEIS, 1935].

$$s = (Q/4\pi T)W(u)$$

$$= (Q/4\pi T)(-0.5772 - \log_e u + u - u^2/2.2! + u^3/3.3! - \ldots) \quad (1)$$

Here $u = r^2S/4Tt$, r = distance from the discharging well, t = time elapsed since start of discharge, T = transmissibility of the aquifer (discharge per unit normal width per unit hydraulic gradient), S = coefficient of storage (volume of water that a unit decline of head releases from storage in a vertical prism of the aquifer of unit cross section), and Q = discharge of the well.

For small values of (r^2/t) compared to the value of $(4T/S)$, u will be so small that the series following the first two terms in the series in equation (1) may be neglected. Thus, where values of (r^2/t) are relatively small, equation (1) may, for all practical purposes, be approximated as in equation (2).

$$s = (Q/4\pi T)[\log_e(1/u) - 0.5772]$$

$$= (Q/4\pi T)[\log_e(4Tt/r^2S) - 0.5772]$$

or

$$s = (Q/4\pi T)\log_e(4e^{-0.5772}Tt/r^2S) = (Q/4\pi T)\log_e(2.25Tt/r^2S) \quad (2)$$

The approximation will be tolerable where u is less than about 0.02. Converting to the common logarithm, we may rewrite equation (2) in any one of the three forms in equations (3), (4), and (5).

$$s = -(2.303Q/2\pi T)[\log_{10}r - (1/2)\log_{10}(2.25Tt/S)] \quad (3)$$

$$s = (2.303Q/4\pi T)[\log_{10}t - \log_{10}(r^2S/2.25T)] \quad (4)$$

$$s = -(2.303Q/4\pi T)[\log_{10}(r^2/t) - \log_{10}(2.25T/S)] \quad (5)$$

The only variables in these equations are the drawdown s, the distance r, and the time t. It is apparent that when t is constant, (3) will be the equation of the straight-line plot of s against $\log_{10}r$. Similarly, when r is constant, (4) will be the equation of the straight-line plot of s against $\log_{10}t$. Moreover, with r and t combined into the single variable (r^2/t), (5) will be the equation of the straight-line plot of s against $\log_{10}(r^2/t)$.

In each equation the slope of the corresponding straight-line plot is represented by the quantity on the outside of the brackets, and the intercept of the straight line on the zero-drawdown line is represented by the second term within the brackets.

As T is the only unknown in the quantity representing the slope, the coefficient of transmissibility is readily determined from a semi-logarithmic plot of observed data by equating the slope of the plot with the corresponding quantity in equation (3), (4), or (5), and solving for T. After T is determined, the only unknown remaining in the term representing the intercept will be S. Therefore, the coefficient of storage may then be determined by equating the intercept of the plot with the corresponding term, and solving for S.

The plots will be straight lines only where (r^2/t) is relatively small so that u is small. A measurement of drawdown that is made too soon after the discharge is begun, or too far from the discharging well, will plot not on the straight line, but on a curve asymptotic to it. However, in tests of artesian aquifers u becomes small soon after the discharge is begun, and hence in most cases little, if any, of the data will fall off the straight line.

The three types of graphs that correspond respectively to equations (3), (4), and (5) may be referred to as the distance-drawdown graph, the time-drawdown graph, and the composite-drawdown graph. The type of graph to be selected for determining the coefficients from a given discharging-well test will depend on the set of data collected in the field.

Distance-drawdown graph--This is a graph of the drawdown at a time t after the discharge begins, plotted against r on semi-logarithmic paper with r on the logarithmic scale. It may be thought of as a radial profile of the (logarithmic) cone of depression. Equating the quantity outside of the brackets in equation (3) with the slope of the graph, $2.303Q/2\pi T = \Delta s/\Delta\log_{10}r$ = slope of plot, whence $T = -(2.303Q/2\pi)(\Delta\log_{10}r/\Delta s)$. The negative sign indicates that s decreases as $\log_{10}r$ increases. For convenience, $\Delta\log_{10}r$ may be made unity by having it represent one logarithmic cycle, whereupon

$$T = -2.303Q/2\pi\Delta s \quad (6)$$

where Δs is the difference in drawdown over one logarithmic cycle.

Equating the second term in brackets in equation (3) with the intercept of the straight line on the zero-drawdown line, and solving for the coefficient of storage, gives equation (7).

$$S = 2.25Tt/r_0^2 \qquad (7)$$

where r_0 is the value of r at the s = 0-intercept.

Figure 1 is a distance-drawdown graph for wells that are 49, 100, and 150 feet from another well discharging at the rate of 2.23 cfs [test by S. W. LOHMAN reported by WENZEL, 1942]. The drawdowns at these distances after 18 days of continuous discharge were 5.09, 4.08, and 3.10 feet, respectively. The difference in drawdown over one logarithmic cycle is (0.69 ft - 4.07 ft) = -3.38 ft. Therefore, from equation (6), T = 2.303(2.23 cfs)/(2 × 3.38 ft) = 0.242 cfs/ft.

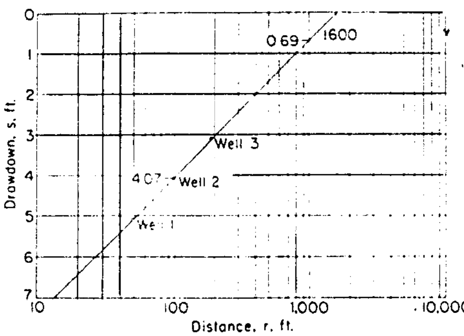

Fig. 1--Distance-drawdown graph based on drawdowns in three wells after 18 days of continuous discharge from an unconfined sand, Q = 2.23 cfs

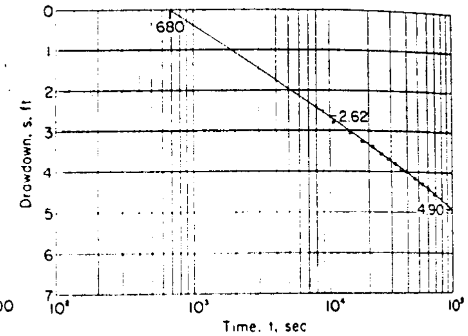

Fig. 2--Time-drawdown graph for a well 1200 feet from another well discharging from a confined sand, Q = 3.00 cfs

The straight line drawn through the plotted points intersects the zero-drawdown line at r_0 = 1600 ft. Thus, from equation (7), S = 2.25(0.242 cfs/ft)(18 days × 86,400 sec/day)/(1600 ft)^2 = 0.33.

Time-drawdown graph--This graph is a plot of the drawdowns in one of the observed wells against t on semi-logarithmic paper, with t on the logarithmic scale. The formulas for T and S are as in equations (8) and (9).

$$T = 2.303Q/4\pi\Delta s \qquad (8)$$

$$S = 2.25Tt_0/r^2 \qquad (9)$$

where t_0 is the value of t at the intercept.

Figure 2 is a time-drawdown graph for a well 1200 feet from another well discharging 3.00 cfs from a confined aquifer [JACOB, 1946]. The plotted points represent water-level readings from an automatic water-stage recording instrument, selected first at one-hour intervals and later at two-hour intervals. The change in drawdown over one logarithmic cycle is 2.28 feet. Accordingly, from equation (8), T = 2.303 (3.00 cfs)/(4π × 2.28 ft) = 0.241 cfs/ft.

The fact that this value for the coefficient of transmissibility agrees closely with that in the preceding example is fortuitous inasmuch as the two sets of data are from tests on different aquifers.

The intercept on the zero-drawdown line is t_0 = 680 seconds. Therefore, from equation (9), S = 2.25 (0.241 cfs/ft)(680 sec)/(1200 ft)^2 = 0.00026.

Composite drawdown graph--This graph is a plot of the drawdowns in several observed wells at different times against (r^2/t), on semi-logarithmic paper. The formulas for the coefficients of transmissibility and storage are as in equations (10) and (11).

$$T = -(2.303Q/4\pi)/\Delta s \qquad (10)$$

$$S = 2.25T/(r^2/t)_0 \qquad (11)$$

where $(r^2/t)_0$ is the value of (r^2/t) at the intercept.

Figure 3 is a composite drawdown graph that includes, in addition to the drawdowns in Figure 2, the drawdowns in a second idle well 1300 feet from the discharging well, and the drawdowns in the discharging well itself. The drawdowns in the discharging well are adjusted for an inferred screen loss of 28.5 feet [JACOB, 1946]. The discharging well is gravel-walled and its screen has a nominal diameter of 18 inches. The effective radius of the well is assumed to be 0.75 foot.

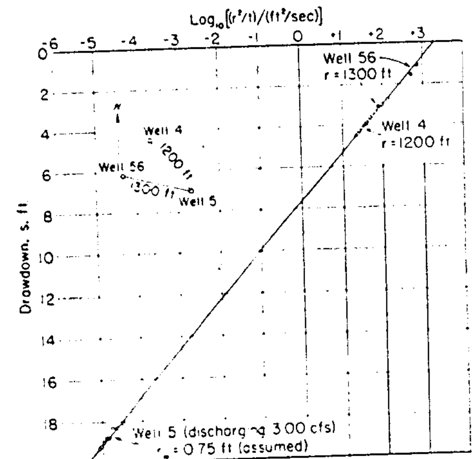

Fig. 3--Composite drawdown graph based on drawdowns observed in a discharging well and two neighboring wells in a confined sand (compare with Fig. 2)

The change in drawdown over one logarithmic cycle is -2.31 feet. This value substituted in equation (10) gives a coefficient of transmissibility of 0.238 cfs/ft. Inasmuch as the measurement of the discharge is correct only to two significant figures, this value does not differ significantly from that determined from Figure 2.

The intercept on the zero-drawdown line is $(r^2/t)_0$ = 2000 sq ft/sec. From this value, the coefficient of storage is computed to be 0.00027, which agrees closely with the value determined from Figure 2.

Generalized straight-line method

Before proceeding with the generalization of the straight-line method, it will be necessary to adopt a set of distinctive symbols to represent the various physical elements involved. The numerals 1, 2, 3, . . . will be used to identify the observed wells, and the letter i will be the general symbol for indicating any one of them. Thus, "Well 1" will be understood to mean Well 1, Well 2, Well 3, etc., in turn. Other symbols are: ΔQ_k = increment of discharge for k = 1, 2, 3, 4, . . . n;

t^k = time elapsed since the inception of ΔQ_k for $t^k = t', t'', t''', t^{iv}, \ldots t^n$; r_{ik} = distance from observed well i to the discharging well in which ΔQ_k occurred; $\Delta s_i{}^k$ = partial drawdown in observed well i produced by the increment of discharge ΔQ_k at the time t^k;

$$Q_n = \Delta Q_1 + \Delta Q_2 + \Delta Q_3 + \ldots \Delta Q_n = \sum_{k=1}^{n} \Delta Q_k$$

which is the algebraic sum of increments of discharge ΔQ_1 to ΔQ_n; and $s_i{}^n$ = total drawdown in observed well i produced by increments of discharge ΔQ_1 to ΔQ_n.

An increment of discharge ΔQ_k may be the initial discharge or a subsequent increase or decrease in discharge in any one of the discharging wells. Increases in discharge will be positive increments, and decreases will be negative. It will be convenient to assign numerals to k in chronological order, but where two or more increments of discharge occur simultaneously, the numerals may be assigned arbitrarily.

In the treatment of problems involving multiple discharging wells, or changes in the discharge of a single well, use is made of the principle of superposition, whereby it is assumed that the total drawdown produced in a given well at a given time by several increments of discharge is the algebraic sum of the drawdowns that would be produced independently by those increments of discharge. So far, the results of discharging-well tests have verified this assumption for artesian conditions.

Equation (12) is according to the principle of superposition.

$$s_i{}^n = \Delta s_i{}' + \Delta s_i{}'' + \Delta s_i{}''' + \ldots \Delta s_i{}^n = \sum_{k=1}^{n} \Delta s_i{}^k \quad (12)$$

From equation (2) the partial drawdown produced in an observed well i by an increment of discharge ΔQ_k is approximately $\Delta s_i{}^k = (2.303\Delta Q_k/4\pi T)\log_{10}(2.25Tt^k/r^2{}_{ik}S)$, and from equation (12) the total drawdown, after n increments of discharge, is in equation (13), for n = 1, 2, 3, etc.

$$s_i{}^n = \sum_{k=1}^{n} \Delta s_i{}^k = \sum_{k=1}^{n} (2.303\Delta Q_k/4\pi T)\log_{10}(2.25Tt^k/r^2{}_{ik}S) \quad (13)$$

Dividing both sides of equation (13) by Q_n, equation (13a) results

$$s_i{}^n/Q_n = \sum_{k=1}^{n} (2.303\Delta Q_k/4\pi TQ_n)\log_{10}(2.25Tt^k/r^2{}_{ik}S) \quad (13a)$$

This may be written as in equation (14) or (15)

$$(s/Q)_i{}^n = -(2.30/4\pi T)[2\sum_{k=1}^{n}(\Delta Q_k/Q_n)\log_{10}r_{ik} - \sum_{k=1}^{n}(\Delta Q_k/Q_n)\log_{10}t^k - \log_{10}(2.25T/S)] \quad (14)$$

$$(s/Q)_i{}^n = -(2.30/4\pi T)[\sum_{k=1}^{n}(\Delta Q_k/Q_n)\log_{10}(r^2/t)_i{}^k - \log_{10}(2.25T/S)] \quad (15)$$

The first and second terms in brackets in equation (14) and the first term in brackets in equation (15) are the logarithms of the weighted logarithmic means of r^2, t, and (r^2/t) respectively. The weighted logarithmic means may be represented by $\bar{r}_{in}$, $\bar{t}^n$, and $\overline{(r^2/t)}_i{}^n$. Substituting these symbols in equations (14) and (15), we may now write the three equations (16), (17), and (18).

$$(s/Q)_i{}^n = -(2.303/2\pi T)[\log_{10}\bar{r}_{in} - (1/2)\log_{10}(2.25T\bar{t}^n/S)] \quad (16)$$

$$(s/Q)_i{}^n = (2.303/4\pi T)[\log_{10}\bar{t}^n - \log_{10}(\bar{r}^2{}_{in}S/2.25T)] \quad (17)$$

$$(s/Q)_i{}^n = -(2.303/4\pi T)[\log_{10}\overline{(r^2/t)}_i{}^n - \log_{10}(2.25T/S)] \quad (18)$$

These equations correspond with equations (3), (4), and (5) for single discharging wells, but include in addition to $\bar{s}_i{}^n$, $\bar{r}_{in}$, and $\bar{t}^n$, a fourth variable, Q_n. So that equations (16), (17), and (18) will be the equations of straight-line plots, Q_n has been combined with $s_i{}^n$ into a single variable $(s/Q)_i{}^n$, which may be referred to as the "specific drawdown" (drawdown per unit discharge). Thus, (16), (17), and (18) are the equations of the straight-line plots of the specific drawdown against $\bar{r}_{in}$, $\bar{t}^n$, and $\overline{(r^2/t)}_i{}^n$, respectively, where $\bar{t}^n$ is constant in equation (16), $\bar{r}_{in}$ is constant in equation (17), and $\bar{r}_{in}$ and $\bar{t}^n$ are combined into a single variable in equation (18). As in equations (3), (4), and (5), the slope of each plot is represented by the quantity on the outside of the brackets in the corresponding equation, and the intercept of the extension of the plot at $(s/Q)_i{}^n = 0$ is represented by the second term within the brackets.

The weighted logarithmic mean distance $\bar{r}_{in}$ for a given observed well at a given time may be computed in the following manner: (1) Multiply each increment of discharge that occurred before the given time by the logarithm of the distance from the observed well to the well in which the increment occurred; (2) sum the products algebraically; (3) divide the sum of the products by the algebraic sum of the increments of discharge; and (4) extract the antilogarithm of the quotient. The result will be the distance $\bar{r}_{in}$. The weighted logarithmic means $\bar{t}^n$ and $\overline{(r^2/t)}_i{}^n$ are computed in a similar manner, but where $\bar{r}_{in}$ and $\bar{t}^n$ are already computed, $\overline{(r^2/t)}_i{}^n$ may be obtained more conveniently by dividing $\bar{r}^2{}_{in}$ by $\bar{t}^n$ directly.

The weighted logarithmic means $\bar{r}_{in}$ and $\bar{t}^n$ both have physical significance. From a comparison of equation (16) with equation (3) it is evident that $\bar{r}_{in}$ is the distance at which a single well discharging at a rate Q_n would produce the drawdown $s_i{}^n$ at the elapsed time $\bar{t}^n$ after the discharge began. A recognition of the significance of these quantities is helpful in interpreting the plots.

The three types of graphs corresponding, respectively, to equations (16), (17), and (18) are referred to as the <u>generalized distance-drawdown graph</u>, the <u>generalized time-drawdown graph</u>, and the <u>generalized composite drawdown graph</u>. The formulas for determining the coefficients of transmissibility and storage from these graphs may be derived in the same manner as in the method for a single well discharging uniformly; that is, by equating the slopes and the intercepts of the plats with the corresponding quantities in the respective equations. The formulas are as in the following paragraphs.

<u>Generalized distance-drawdown graph</u>

$$T = -2.303/[2\pi\Delta(s/Q)_i{}^n] \quad (19)$$

where $\Delta(s/Q)_i{}^n$ is the change in specific drawdown over one logarithmic cycle.

$$S = 2.25T\bar{t}_0/\bar{r}^2{}_{in} \quad (20)$$

where $\bar{r}_0$ is the value of $\bar{r}_{in}$ at the intercept.

<u>Generalized time-drawdown graph</u>

$$T = 2.303/[4\pi\Delta(s/Q)_i{}^n] \quad (21)$$

$$S = 2.25T\bar{t}_0/\bar{r}^2{}_{in} \quad (22)$$

where $\bar{t}_0$ is the value of $\bar{t}^n$ at the intercept.

<u>Generalized composite drawdown graph</u>

$$T = -2.303/[4\pi\Delta(s/Q)_i{}^n] \quad (23)$$

$$S = 2.25T/\overline{(r^2/t)}_0 \quad (24)$$

where $\overline{(r^2/t)}_0$ is the value of $\overline{(r^2/t)}_i{}^n$ at the intercept. The use of the generalized composite drawdown graph is demonstrated in the example that follows.

Figure 4(a) shows the locations of wells at the Central Plant of the municipal water supply of Houston, Texas [GUYTON and ROSE, 1945]. The columnar sections, based on well logs, show by stippling the sands penetrated by the wells. The positions of the well screens are also indicated.

Figure 4(b) is a graph of the drawdown and subsequent partial recovery observed in Well F5 on October 10, 1939 [JACOB, 1941]. Well F10, 850 feet from Well F5, began pumping 2.27 cfs at 10^h00^m and stopped pumping at 18^h45^m. Well F1, 780 feet away, began pumping 2.79 cfs at 10^h30^m and stopped pumping at 20^h05^m. Well F12, 1060 feet away, began pumping 3.56 cfs at 11^h00^m and continued pumping through the end of the test. Measurements of the water level in Well F5 were made throughout the day. Some of these measurements, expressed as drawdowns, are platted in Figure 4(b), where the measurements used in applying the generalized straight-line graphical method are plotted each as two concentric circles.

Computations to determine values of the weighted logarithmic mean $\overline{(r^2/t)}^n$ and the corresponding values of the specific drawdown $(s/Q)^n$ are given in Table 1. (The subscript i, which refers to

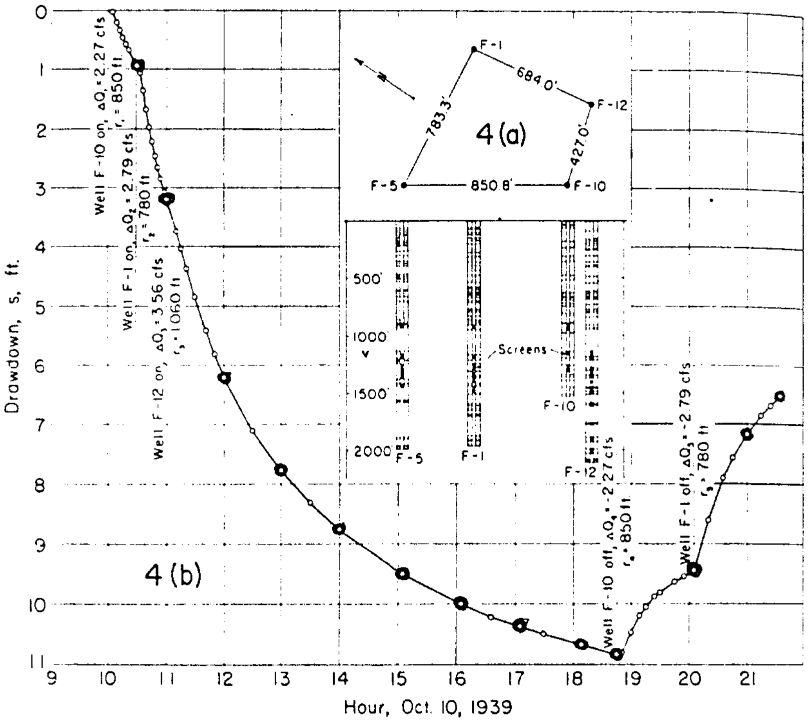

Fig. 4--(a) Map showing relative location of wells at Central Plant, Houston, Texas, and columnar sections based on well logs (after GUYTON and ROSE)
(b) Drawdown and subsequent partial recovery observed in Well F5, October 10, 1939, resulting from staggered operation of wells F10, F1, and F12

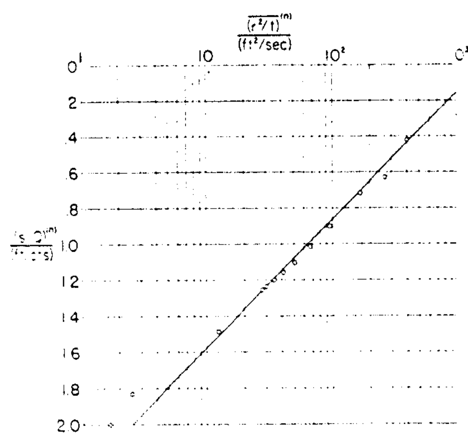

Fig. 5--Generalized composite drawdown graph for Well F5, Central Plant, Houston, Texas, October 10, 1939

Table 1--Computations of specific drawdown and weighted logarithmic mean $\overline{(r^2/t)}^n$ for Well F5, Central Plant, Houston, Texas, October 10, 1939

Time	k	n	Dis-charge well	r_k	t^k	(r^2_k/t^k)	$Log_{10}(r^2_k/t^k)$	ΔQ_k	(9)×(8)	$Log_{10}\overline{(r^2/t)}^n$	$\overline{(r^2/t)}^n$	s^n	$(s/Q)^n$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
h m				ft	sec	ft²/sec	ft²/sec	cfs	cfs		ft²/sec	ft	ft/cfs
10 30	1	1	F10	850	1800	402	2.604	2.27	....	2.604	402	0.96	0.423
11 00	1	..	F10	850	3600	201	2.303	2.27	5.23	....	....	...	....
	2	..	F1	780	1800	338	2.529	2.79	7.06	....	....	...	....
		2	...	...	...	...	....	5.06	12.29	2.429	269	3.20	0.632
12 00	1	..	F10	850	7200	100.4	2.002	2.27	4.54	....	....	...	....
	2	..	F1	780	5400	112.6	2.052	2.79	5.73	....	....	...	....
	3	..	F12	1060	3600	312	2.494	3.56	8.88	....	....	...	....
		3	...	...	...	...	....	8.62	19.15	2.222	167	6.21	0.720
13 00	1	..	F10	850	10800	66.9	1.826	2.27	4.15	....	....	...	....
	2	..	F1	780	9000	67.6	1.830	2.79	5.11	....	....	...	....
	3	..	F12	1060	7200	156	2.194	3.56	7.81	....	....	...	....
		3	...	...	...	...	....	8.62	17.07	1.980	95.5	7.77	0.901
14 00	1	..	F10	850	14400	50.2	1.701	2.27	3.86	....	....	...	....
	2	..	F1	780	12600	48.3	1.684	2.79	4.70	....	....	...	....
	3	..	F12	1060	10800	104	2.017	3.56	7.18	....	....	...	....
		3	...	...	...	...	....	8.62	15.74	1.826	67.0	8.76	1.016
15 05	1	..	F10	850	18300	39.5	1.597	2.27	3.63	....	....	...	....
	2	..	F1	780	16500	36.9	1.567	2.79	4.37	....	....	...	....
	3	..	F12	1060	14700	76.4	1.883	3.56	6.70	....	....	...	....
		3	...	...	...	...	....	8.62	14.70	1.705	50.7	9.50	1.102
16 05	1	..	F10	850	21900	33.0	1.518	2.27	3.45	....	....	...	....
	2	..	F1	780	20100	30.3	1.481	2.79	4.13	....	....	...	....
	3	..	F12	1060	18300	61.4	1.788	3.56	6.37	....	....	...	....
		3	...	...	...	...	....	8.62	13.95	1.618	41.5	10.00	1.160
17 05	1	..	F10	850	25500	28.3	1.453	2.27	3.30	....	....	...	....
	2	..	F1	780	23700	25.7	1.410	2.79	3.93	....	....	...	....
	3	..	F12	1060	21900	51.3	1.710	3.56	6.09	....	....	...	....
		3	...	...	...	...	....	8.62	13.32	1.545	35.1	10.37	1.203
18 08	1	..	F10	850	29280	24.7	1.392	2.27	3.160	....	....	...	....
	2	..	F1	780	27480	22.1	1.345	2.79	3.753	....	....	...	....
	3	..	F12	1060	25680	43.8	1.641	3.56	5.842	....	....	...	....
		3	...	...	...	...	....	8.62	12.755	1.4797	30.18	10.67	1.238
18 45	1	..	F10	850	31500	22.9	1.361	2.27	3.089	....	....	...	....
	2	..	F1	780	29700	20.5	1.311	2.79	3.658	....	....	...	....
	3	..	F12	1060	27900	40.3	1.605	3.56	5.714	....	....	...	....
		3	...	...	...	...	....	8.62	12.461	1.4456	27.90	10.84	1.258
20 05	1	..	F10	850	36300	19.9	1.299	2.27	2.949	....	....	...	....
	2	..	F1	780	34500	17.6	1.246	2.79	3.476	....	....	...	....
	3	..	F12	1060	32700	34.4	1.536	3.56	5.468	....	....	...	....
	4	..	F10	850	4800	150.5	2.177	-2.27	-4.942	....	....	...	....
		4	...	...	...	...	....	6.35	6.951	1.0946	12.43	9.45	1.488
21 00	1	..	F10	850	39600	18.2	1.261	2.27	2.862	....	....	...	....
	2	..	F1	780	37800	16.1	1.207	2.79	3.368	....	....	...	....
	3	..	F12	1060	36000	31.2	1.494	3.56	5.319	....	....	...	....
	4	..	F10	850	8100	89.2	1.950	-2.27	-4.427	....	....	...	....
	5	..	F1	780	3300	184.4	2.266	-2.79	-6.322	....	....	...	....
		5	...	...	...	...	....	3.56	0.800	0.2247	1.678	7.16	2.011
21 35	1	..	F10	850	41700	17.3	1.239	2.27	2.813	....	....	...	....
	2	..	F1	780	39900	15.2	1.183	2.79	3.301	....	....	...	....
	3	..	F12	1060	38100	29.5	1.470	3.56	5.233	....	....	...	....
	4	..	F10	850	10200	70.8	1.850	-2.27	-4.199	....	....	...	....
	5	..	F1	780	5400	112.7	2.052	-2.79	-5.725	....	....	...	....
		5	...	...	...	...	....	3.56	1.423	0.3997	2.51	6.51	1.829

Note: The subscript i, which refers to the observation well, is omitted, because only one observation well is involved in the example.

the observation well, is omitted from the symbols because only one observation well is involved in the example.) The computation procedure may be observed by following the headings of the columns in the Table. The increments of discharge that occurred before the time given in column (1) are listed and summed algebraically in column (9). These increments of discharge are multiplied by the logarithms of the corresponding values of (r^2/t), and the products are listed and summed algebraically in column (10). The sum of the products given in column (10) is then divided by the sum of the increments of discharge given in column (9), and the quotient is listed in column (11). The antilogarithm of this quotient, listed in column (12) is the weighted logarithmic mean $\overline{(r^2/t)^n}$. The corresponding value of the specific drawdown $(s/Q)^n$ is listed in column (14).

The data given in columns (12) and (14) are plotted in Figure 5. The alignment of the plotted points is not bad in view of the fact that the screens of the four wells are set at various depths and also the fact that the water-bearing sands are lenticular and vary in thickness and permeability from one well to another. The extent to which these or other circumstances might vitiate the method used may be judged most readily from the alignment of the points on a simple, straight-line graph such as Figure 5.

The change in specific drawdown $\Delta(s/Q)^n$ over one logarithmic cycle is -0.71 ft per cfs. Therefore, from equation (23) $T = 2.303/(4\pi \times 0.71 \text{ ft/cfs}) = 0.26$ cfs/ft.

The extension of the straight line in Figure 5 intersects the line of zero drawdown at $\overline{(r^2/t)^n} = \overline{(r^2/t)_0}^n = 1650$ ft²/sec. Thus, from equation (24) $S = 2.25(0.26 \text{ cfs/ft})/(1650 \text{ ft}^2/\text{sec}) = 0.00035$.

References

GUYTON, W. F., and ROSE, N. A., Quantitative studies of some artesian aquifers in Texas, Econ. Geol., pp. 193-226, May, 1945.

JACOB, C. E., On the flow of water in an elastic artesian aquifer, Trans. Amer. Geophys. Union, v. 21, pp. 574-586, 1940.

JACOB, C. E., Coefficients of storage and transmissibility obtained from pumping tests in the Houston district, Texas, Trans. Amer. Geophys. Union, v. 22, pp. 744-756, 1941.

JACOB, C. E., Drawdown test to determine screen loss and effective radius of an artesian well, to be published in Proc. Amer. Soc. Civil Engrs., 1946.

THEIS, C. V., The relation between the lowering of the piezometric surface and the rate and duration of discharge of a well using ground-water storage, Trans. Amer. Geophys. Union, v. 16, pp. 519-524, 1935.

WENZEL, L. K., Methods for determining permeability of water-bearing materials, U. S. Geol. Surv. W.-S. Paper 887, 1942.

WENZEL, L. K., and GREENLEE, A. L., A method for determining transmissibility and storage coefficients by tests of multiple well systems, Trans. Amer. Geophys. Union, v. 25, pp. 547-560, 1944.

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NOTES ON COOPER and JACOB (1946)

DEVELOPMENT OF EQUATIONS (14) ÷ (18)

STARTING FROM (13) :

$$s_i^n = \sum_{k=1}^{n} \Delta s_i^k = \sum_{k=1}^{n} \left(2.303 \, \Delta Q_k / 4\pi T\right) \log_{10}\left(2.25 T t^k / r_{ik}^2 S\right)$$

DIVIDE BOTH SIDES BY Q_n :

$$\frac{s_i^n}{Q_n} = \sum_{k=1}^{n} \left(2.303 \frac{\Delta Q_k}{Q_n} / 4\pi T\right) \log_{10}\left(2.25 T t^k / r_{ik}^2 S\right) \quad \text{—(13a)}$$

$$= \frac{2.303}{4\pi T} \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log_{10}\left(\frac{2.25 T t^k}{r_{ik}^2 S}\right)$$

EXPANDING THE LOG TERM :

$$\frac{s_i^n}{Q_n} = \frac{2.303}{4\pi T} \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log_{10}\left(\frac{2.25T}{S} \cdot t^k \cdot \frac{1}{r_{ik}^2}\right) \quad \text{—(13*)}$$

(1) Working in terms of separate t and r :

Expand (13*) as :

$$\frac{s_i^n}{Q_n} = \frac{2.303}{4\pi T} \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \left[\log_{10}\left(\frac{2.25T}{S}\right) + \log_{10}\left(t^k\right) - 2\log_{10}\left(r_{ik}\right)\right]$$

Expanding more :

$$\frac{s_i^n}{Q_n} = \frac{2.303}{4\pi T}\left(\sum_{k=1}^{n}\frac{\Delta Q_k}{Q_n}\log_{10}\left(\frac{2.25T}{S}\right) + \sum_{k=1}^{n}\frac{\Delta Q_k}{Q_n}\log_{10}\left(t^k\right) - 2\sum_{k=1}^{n}\frac{\Delta Q_k}{Q_n}\log_{10}\left(r_{ik}\right)\right)$$

The first is simplified as follows :

$$\sum_{k=1}^{n}\frac{\Delta Q_k}{Q_n}\log_{10}\left(\frac{2.25T}{S}\right) = \log_{10}\left(\frac{2.25T}{S}\right)\sum_{k=1}^{n}\frac{\Delta Q_k}{Q_n}$$

$$= \log_{10}\left(\frac{2.25T}{S}\right)\cdot\frac{1}{Q_n}\sum_{k=1}^{n}\Delta Q_k$$

but $\sum_{k=1}^{n}\Delta Q_k \equiv Q_n$

$$\therefore \sum_{k=1}^{n}\frac{\Delta Q_k}{Q_n}\log_{10}\left(\frac{2.25T}{S}\right) = \log_{10}\left(\frac{2.25T}{S}\right)$$

$$\therefore \frac{s_i^n}{Q_n} = \frac{2.303}{4\pi T}\left(\log_{10}\left(\frac{2.25T}{S}\right) + \sum_{k=1}^{n}\frac{\Delta Q_k}{Q_n}\log_{10}\left(t^k\right) - 2\sum_{k=1}^{n}\frac{\Delta Q_k}{Q_n}\log_{10}\left(r_{ik}\right)\right) \quad (14)$$

② Working in terms of r^2/t :

Expand (13*) as :

$$\frac{s_i^n}{Q_n} = \frac{2.303}{4\pi T} \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \left[\log_{10}\left(\frac{2.25T}{S}\right) - \log_{10}\left(\frac{r^2_{ik}}{t^k}\right) \right]$$

$$= \frac{2.303}{4\pi T} \left(\sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log_{10}\left(\frac{2.25T}{S}\right) - \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log_{10}\left(\frac{r^2_{ik}}{t^k}\right) \right)$$

$$= \frac{2.303}{4\pi T} \left(\log_{10}\left(\frac{2.25T}{S}\right) - \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log_{10}\left(\frac{r^2_{ik}}{t^k}\right) \right)$$

—(15)

③ Working in terms of t "alone" [for one pumping-observation well pair, for example]

Expand (13*) as :

$$\frac{s_i^n}{Q_n} = \frac{2.303}{4\pi T} \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \left[\log_{10}\left(\frac{2.25T}{S r^2_{ik}}\right) + \log_{10}\left(t^k\right) \right]$$

$$= \frac{2.303}{4\pi T} \left(\sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log_{10}\left(\frac{2.25T}{S r^2_{ik}}\right) + \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log_{10}\left(t^k\right) \right)$$

—(15*)

④ Working in terms of r "alone"

[for one pumping well and a set of observation wells, all at one time, for example]

Expand (13*) as:

$$\frac{s_i^n}{Q_n} = \frac{2.303}{4\pi T} \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \left[\log_{10}\left(\frac{2.25 T t^k}{S}\right) - 2 \log_{10}(r_{ik}) \right]$$

$$= \frac{2.303}{4\pi T} \left(\sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log_{10}\left(\frac{2.25 T t^k}{S}\right) - 2 \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log_{10}(r_{ik}) \right) \quad \text{——(15**)}$$

DEFINE WEIGHTED LOGARITHMIC MEANS:

A) $$\log_{10} \bar{t}^n = \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log_{10}(t^k)$$

B) $$\log_{10} \bar{r}_{in} = \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log_{10}(r_{ik})$$

C) $$\log_{10} \overline{\left(\frac{r_i^2}{t}\right)}^n = \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log_{10}\left(\frac{r_{ik}^2}{t^k}\right)$$

1. SUBSTITUTING Ⓐ and Ⓑ IN (14) YIELDS:

$$\frac{s_i^n}{Q_n} = \frac{2.303}{4\pi T}\left(\log_{10}\left(\frac{2.25T}{S}\right) + \log_{10}\bar{t}_n - 2\log_{10}\bar{r}_{in}\right)$$

2. SUBSTITUTING Ⓒ IN (15) YIELDS:

$$\frac{s_i^n}{Q_n} = \frac{2.303}{4\pi T}\left(\log_{10}\left(\frac{2.25T}{S}\right) - \log_{10}\overline{\left(\frac{r_i^2}{t}\right)^n}\right) \quad \text{—(18)}$$

3. SUBSTITUTING Ⓐ IN (15*) YIELDS:

$$\frac{s_i^n}{Q_n} = \frac{2.303}{4\pi T}\left(\sum_{k=1}^{n}\frac{\Delta Q_k}{Q_n}\log_{10}\left(\frac{2.25T}{Sr_{ik}^2}\right) + \log_{10}\bar{t}^n\right)$$

4. SUBSTITUTING Ⓑ IN (15**) YIELDS:

$$\frac{s_i^n}{Q_n} = \frac{2.303}{4\pi T}\left(\sum_{k=1}^{n}\frac{\Delta Q_k}{Q_n}\log_{10}\left(\frac{2.25Tt^k}{S}\right) - 2\log_{10}\bar{r}_{in}\right)$$

$$= \frac{2.303}{2\pi T}\left(\frac{1}{2}\sum_{k=1}^{n}\frac{\Delta Q_k}{Q_n}\log_{10}\left(\frac{2.25Tt^k}{S}\right) - \log_{10}\bar{r}_{in}\right)$$

Q : Following "my" definitions of $\log_{10} \bar{t}^n$ and $\log_{10} \bar{r}_{in}$, can it be shown my (3) and (4) are equivalent to Cooper and Jacob's (17) and (16), respectively?

1. $$\log_{10}\left(\frac{2.25T\bar{t}_n^*}{S}\right) \overset{?}{=} \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log_{10}\left(\frac{2.25Tt^k}{S}\right)$$

$$= \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n}\left[\log_{10}\left(\frac{2.25T}{S}\right) + \log_{10}\left(t^k\right)\right]$$

$$= \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log_{10}\left(\frac{2.25T}{S}\right) + \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log_{10}\left(t^k\right)$$

$$= \log_{10}\left(\frac{2.25T}{S}\right) + \log_{10} \bar{t}^n$$

$$= \log_{10}\left(\frac{2.25T\bar{t}^n}{S}\right) \quad \checkmark$$

2. $$-\log_{10}\left(\frac{\bar{r}_{in}^2 S}{2.25T}\right) \overset{?}{=} \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log_{10}\left(\frac{2.25T}{S r_{ik}^2}\right)$$

$$= \sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n}\left[\log_{10}\left(\frac{2.25T}{S}\right) - 2\log_{10}\left(r_{ik}^2\right)\right]$$

$$= \log_{10}\left(\frac{2.25T}{S}\right) - 2\sum_{k=1}^{n} \frac{\Delta Q_k}{Q_n} \log\left(r_{ik}\right)$$

$$= \log_{10}\left(\frac{2.25T}{S}\right) - 2\log_{10} \bar{r}_{in}$$

$$= -\log_{10}\left(\frac{\bar{r}_{in}^{2}\, S}{2.25T}\right)$$ ✓

A : Yes. And therefore :

3. $$\frac{s_i^n}{Q_n} = \frac{2.303}{4\pi T}\left(-\log_{10}\left(\frac{\bar{r}_{in}^{2}\, S}{2.25T}\right) + \log_{10}\bar{t}^{n}\right)$$ —(17)

4. $$\frac{s_i^n}{Q_n} = \frac{2.303}{2\pi T}\left(\frac{1}{2}\log_{10}\left(\frac{2.25T\,\bar{t}^{n}}{S}\right) - \log_{10}\bar{r}_{in}\right)$$ —(16)